Essays in Asset Pricing with Search Frictions

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Introduction

My dissertation is in theoretical asset pricing and the general question I am trying to address is: How do trading and informational frictions affect asset returns and the allocative efficiency of markets?
I have been working mainly on the "search frictions" that prevail on over-the-counter (OTC) markets. *Search frictions* refer to the necessity to search for counter-parties when trading on a bilateral market. This search process is time-consuming and of uncertain duration. As a result, and as shown in the seminal contribution Duffie, Gârleanu, and Pedersen [2005], search frictions lead to an inefficient asset allocation and affect prices. OTC markets are huge and include essentially all fixed income securities, a vast majority of derivatives, real estate, and foreign exchange markets.

In the three chapters that my dissertation comprises, I analyze three specific issues related to search frictions. In the first chapter, I evaluate the cross-market effects of search frictions. More specifically, I study a general equilibrium model in which investors face endowment risk and trade two correlated assets; one asset is traded on a liquid market whereas the other is traded on an illiquid over-the-counter (OTC) market. Endowment shocks not only make prices drop, they also make the OTC asset more difficult to sell, creating an endogenous liquidity risk. This liquidity risk increases the risk premium of both the OTC asset and liquid asset. Furthermore, the OTC market frictions increase the trading volume and the cross-sectional dispersion of ownership in the liquid market. Finally, if the economy starts with only the OTC market, then I explain how opening a correlated liquid market can increase or decrease the OTC price depending on the illiquidity level. The model's predictions can help explain several empirical findings.

In the second chapter of my thesis, I study how investors balance two types of illiquidity when choosing an optimal portfolio. The two types of illiquidity are search frictions, the risk of being sometimes unable to trade, and transaction costs, the costly opportunity to trade at any time. More specifically, I study an equilibrium model in which investors hedge endowment risk on two markets. The first market is centralized and trades can be executed at any time but against proportional transaction costs. The second market is OTC and trading requires to search for a counter-party. I show the existence of an equilibrium for any level of transaction costs and characterize in closed-form how the two frictions jointly modify the risk-premia on both markets. The transaction costs reduce both the transaction sizes and frequencies on the centralized market, and affect the price on the OTC market by making hedging more costly. The prices bargained OTC depend non-monotonically on the transaction costs on the exchange. I link my results to recent regulatory proposals and show how taxing CDS trading may both increase of decrease the cost of debt financing.

In the third and last chapter, which is joint work with Julien Cujean,[1] we evaluate the joint effect of search frictions and asymmetric information on the allocative efficiency of a market and the cost of providing liquidity. More specifically, we study how transparency, modeled as information about one's counterparty liquidity needs, affects the functioning of an OTC market. In our model, investors hedge endowment risk by trading bilaterally in a search-and-matching environment. We construct a bargaining procedure that accommodates information asymmetry regarding investors' inventories. Both the trade size and the trade price are endogenously determined. Increased transparency improves the allocative efficiency of the market. However, it simultaneously increases inventory costs, and leads to a higher cross-sectional dispersion of transaction prices. For investors with large risk exposure, the increase of the inventory costs dominates the benefits of the market efficiency. We link the model's predictions to recent empirical findings regarding the effect of the TRACE reporting system on bond market liquidity.

Key words: Asset Pricing Theory; Over-the-Counter Markets; Search Frictions; General Equilibrium; Information Asymmetry

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Introduction (Français)

Ma thèse est en théorie de la valuation des actifs et j'étudie comment les frictions dans l'exécution des transactions et les frictions d'asymétrie information modifient les rendements des actifs et l'efficience allocative des marchés.
Je me suis concentré sur les "frictions de recherche" (*search frictions* en anglais) que l'on rencontre sur les marchés de gré à gré (*over-the-counter markets* ou *OTC markets* en anglais). Les *frictions de recherche* sont le temps nécessaire à trouver une contrepartie lorsque l'on souhaite exécuter une transaction sur un marché bilatéral. Cette recherche est coûteuse en temps et de durée incertaine. Comme montré dans une influente publication de Duffie et al. [2005], il suit que les frictions de recherche induisent une allocation des actifs qui est inefficiente et modifie les prix de transaction. Les marchés de gré à gré sont énormes et comprennent essentiellement tous les marchés obligataires, une large majorité des marchés de dérivés, l'immobilier, et les marchés de devises.

Dans les trois chapitres des ma thèse, je considère trois questions liées aux frictions de recherche. Dans le premier chapitre, j'analyse l'effet transversal des frictions de recherche à travers différents marchés. Pour être plus spécifique, j'étudie un modèle d'équilibre général dans lequel des investisseurs mitigent leur risque de dotation à l'aide de deux actifs dont les fondamentaux sont corrélés. Un des actifs s'échange sur un marché liquide alors que le second s'échange sur un marché de gré à gré illiquide. Les chocs de dotation ne font pas seulement chuter le prix des actifs, ils rendent aussi la liquidation de l'actif échangé de gré à gré plus difficile, ce qui induit de manière endogène un risque de liquidité. Le risque de liquidité augmente la prime de risque de l'actif illiquide, mais aussi celle de l'actif liquide. De plus, les frictions de recherche augmentent le volume d'échange et la dispersion transversale des positions choisies sur le marché liquide. Finalement, si l'économie débute uniquement avec un marché de gré à gré, je montre comment l'ouverture du marché liquide peut, selon le niveau des frictions, augmenter ou réduire le prix de l'actif illiquide. Les prédictions du modèle peuvent nous aider à expliquer plusieurs résultats empiriques documentés dans la littérature.

Dans le deuxième chapitre de ma thèse, j'étudie comment des investisseurs choisissent leur portefeuille lorsqu'ils font face à deux types d'illiquidité. Ces deux types d'illiquidités sont les frictions de recherche, soit le risque de ne pas toujours être capable d'échanger un actif, et des coûts de transaction, soit la possibilité d'échanger un actif à tout moment mais contre le paiement d'une commission. Pour être plus spécifique, j'étudie un modèle dans lequel des investisseurs mitigent leur risque de dotation sur deux marchés. Le premier marché est une bourse sur laquelle des transactions peuvent être exécutées à tout moment mais contre le paiement d'une commission proportionnelle au montant échangé. Le second marché est un marché de gré à gré sujet à des frictions de recherche. Je montre l'existence et l'unicité d'un équilibre pour n'importe quel niveau de coûts de transactions et caractérise en forme analytique comment les deux frictions se combinent pour modifier la prime de risque sur les deux marchés. Les coûts de transaction réduisent tant la taille que la fréquence des transactions et modifient les prix négociés de gré à gré en rendant la mitigation des risques sur l'autre marché plus coûteuse. Les prix négociés sur le marché de gré à gré dépendent de manière non monotone des coûts de transactions pratiqués sur la bourse. Je tire des parallèles entre les prédictions de mon modèle et des propositions récentes concernant la régulation des marchés financiers. Je montre, par exemple, comment une taxation des transactions sur les dérivés de crédits (*credit default swaps*) pourrait augmenter ou réduire le coût du financement obligataire.

Le troisième et dernier chapitre de ma thèse est un travail commun avec Julien Cujean.[2] Dans ce chapitre, nous évaluons l'effet conjoint des frictions de recherche et de l'asymétrie d'information sur l'efficience de l'allocation des actifs sur un marché de gré à gré et sur les coût induits par une activité d'intermédiation. Pour être plus précis, nous étudions comment la transparence, modélisée comme une information concernant les besoin de liquidité de sa contrepartie, détermine le fonctionnement d'un marché de gré à gré. Dans notre modèle, des investisseurs mitigent leur risque de dotation en échangeant un actif sur un marché bilatéral affecté par des frictions de recherche. Nous proposons une procédure de négociation (*bargaining*) qui prend en compte l'asymétrie d'information concernant les expositions des investisseurs. Tant la quantité échangée que le prix négocié sont déterminés de manière endogène. Un transparence accrue améliore l'efficience de l'allocation de l'actif. Toutefois, cette transparence accrue augmente également les coûts d'inventaires et induit une plus grande dispersion transversale des prix de transactions. Pour les investisseurs avec une large exposition à l'actif, l'accroissement des coûts d'inventaires domine les bénéfices induits par les gains d'efficience. Nous tirons des parallèles entre les prédictions du modèle et des résultats empiriques récents concernant le système TRACE et son effet sur la liquidité des marchés obligataires américains.

Mots-clés: théorie de la valuation d'actif, marchés de gré à gré, frictions de recherche, équilibre général, asymétrie d'information

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Contents

1 Equilibrium Asset Pricing with both Liquid and Illiquid Markets

I[1] study a general equilibrium model in which investors face endowment risk and trade two correlated assets; one asset is traded on a liquid market whereas the other is traded on an illiquid over-the-counter (OTC) market. Endowment shocks not only make prices drop, they also make the OTC asset more difficult to sell, creating an endogenous liquidity risk. This liquidity risk increases the risk premium of both the OTC asset and liquid asset. Furthermore, the OTC market frictions increase the trading volume and the cross-sectional dispersion of ownership in the liquid market. Finally, if the economy starts with only the OTC market, then I explain how opening a correlated liquid market can increase or decrease the OTC price depending on the illiquidity level. The model's predictions can help explain several empirical findings.

1.1 Introduction

Today, many assets are traded increasingly, or even exclusively, in over-the-counter (OTC) markets. For example, essentially all fixed income securities and a vast majority of all existing derivatives are traded OTC. Even for liquid stocks, large block trading among financial

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institutions is typically done off-exchange. Trading in OTC markets requires searching for a suitable counter-party and bargaining over the exact terms of the transaction. In an important contribution, Duffie et al. [2005] model the functioning of an OTC market and show how search frictions affect prices and make the equilibrium allocation of the asset inefficient. A logical question follows as to whether investors hedge the inefficiently allocated asset by trading more liquid instruments with correlated cash-flows. In this paper, I address this question and show how illiquidity due to search frictions spills over to liquid markets, affecting holdings, trading volumes, and risk premia.

Prime examples of securities offering exposure to the same fundamentals but with different levels of liquidity are bonds and credit default swaps (CDSs). Both bonds and CDSs are traded OTC, but CDSs are typically more liquid.[2] Further examples include mortgages and collateralized debt obligations (CDOs), CDSs and index CDSs, options and index options, forwards and futures contracts, and real estate assets and property total return swaps. In all of these pairs, the first asset is less liquid and traded OTC.

The possibility of trading liquid securities may have a non-trivial effect on the transactions bargained on the OTC market. At the individual level, it improves each investor's outside option by increasing the set of investment opportunities. However, it also improves the outside options of an investor's trading counter-parties, leading to equilibrium feed-back effects on the bargaining outcome. At the same time, the search friction on the OTC market creates demand for the liquid asset that is not driven by its fundamentals, leading to illiquidity spillovers. The effect of these spillovers on the equilibrium prices of both the liquid and illiquid assets is a priori unclear and can only be quantified in a general equilibrium model. In this paper, I propose and analyze such a model.

I consider an economy in which risk-averse investors share endowment risks by trading two imperfectly correlated assets. The first is traded in a frictionless way on an exchange, whereas the second is traded on an illiquid OTC market.[3] I adopt the framework developed by Duffie

[2] A number of features make CDSs more liquid than bonds. First, a CDS contract on a bond issuer offer a homogeneous alternative to a possibly very fragmented bond market. Second, CDSs are derivatives and, as a result, new contracts can be created when needed and there is no need to locate an existing contract. Longstaff, Mithal, and Neis [2005] further discuss the relative liquidity of CDS and bond markets. They argue that the frictions on the CDS market are negligible when compared to those on the bond market. Another important difference is that trading a CDSs is far less capital intensive than trading bonds. This difference in the margin requirements increases CDS trading volumes at the expenses of bond trading, which makes CDS trading even cheaper when compared to bond trading. The difference in margin requirements is the object of, for instance, Basak and Croitoru [2000] and Gârleanu and Pedersen [2011], and is not explicitly modeled in this paper.

[3] In my model, there is a dichotomy between the (perfectly) liquid and illiquid markets. This is for ease of exposition and my conclusions are also relevant for pairs of markets that are "unequally illiquid". Again considering the CDS-bond pair, it is true that CDS markets are not perfectly liquid. For example, Bongaerts, de Jong, and Driessen [2011] and Junge and Trolle [2013] find that illiquidity is priced on CDS markets. Still, CDS markets are usually liquid when compared to the underlying bond market, and investors typically hedge a bond portfolio with

et al. [2005] to model the OTC market. In this framework, investors are matched randomly over time and the Nash bargaining solution characterizes the bilateral trades. Illiquidity, measured by the expected search time between two meetings, affects investors both because contacting trading partners is time consuming and because prices are not competitive.

Investors on the OTC market use the liquid asset as an imperfect substitute for the illiquid asset. Specifically, investors hedge their temporary sub-optimal exposure with the liquid instrument.[4] Due to this effect, the trading volume on the liquid market is always higher in the presence of the OTC market. Interestingly, the strictly positive increase in the trading volume sometimes persists even in the limit of vanishing search frictions. I also show that the dispersion of holdings in the liquid asset increases in the illiquidity of the OTC market. This prediction is consistent with the recent empirical findings of Oehmke and Zawadowski [2013] regarding bond and CDS markets. Oehmke and Zawadowski [2013] document an average net exposure on the CDS market that increases in the illiquidity of the bond market.

I now discuss how the frictions on the OTC market affect the expected returns of the liquid asset. I explicitly characterize the equilibrium response of the liquid asset to the frictions on the OTC market, and show that the spillover effect is driven by illiquidity *risk* and not by the illiquidity *level* alone. The mechanism is as follows: An aggregate shock to investors' hedging demand changes the imbalance between buyers and sellers on the OTC market. This imbalance determines the rate at which the illiquid asset is reallocated. The endogenous relationship between the preference shocks and the search friction makes the allocative efficiency of the OTC market time-varying. In equilibrium, agents require a premium for taking exposure to this non-fundamental risk and the correlation between the efficiency of the OTC market and the returns on the liquid asset is priced. When the risk profiles of the two assets are sufficiently similar, the allocative efficiency is positively correlated to the returns of the liquid asset. This means that the liquid asset performs poorly exactly when liquidating one's illiquid portfolio becomes more difficult, and this command a positive risk premium on the liquid asset. This model prediction is consistent with the conclusions of several empirical studies. For example, both Tang and Yan [2006] and Lesplingart, Majois, and Petitjean (2012) show that CDS spreads increase with the illiquidity of the underlying bonds. Das and Hanouna [2009] show how these same CDS spreads increase with the illiquidity of the debt issuer's stock.[5] The pricing of illiquidity risk also has interesting connections with the literature on

CDS contracts and not the other way around.

[4]This type of behavior is not restricted to OTC markets. For example, stock index futures may be traded instead of re-balancing a diversified stock portfolio. Even if each stock is traded on an exchange and relatively liquid, trading one liquid futures is faster, less costly, and more convenient than re-balancing a diversified stock portfolio. Also, exchange-traded funds (ETFs) give investors the opportunity to conveniently adjust their exposures to stocks, bonds, commodities, or currencies. See Greenwich [May 2012] for a survey documenting the use of ETFs as an alternative to trading the underlying market.

[5]When comparing these empirical studies to the model's predictions, the returns on the liquid asset should be

long-run risk, pioneered by Bansal and Yaron [2004]. Namely, as it may take a long time for the market to recover after a liquidity shock, illiquidity leads to *endogenous long run risk* in the economy. This long run risk is priced, and its price depends on the long run value of future liquidity for an investor, expressed by the corresponding certainty equivalent.

Conversely, the liquid market has two effects on the OTC market. On the one hand, it mitigates the search frictions and reduces the illiquidity discount, pushing the price on the OTC market up. On the other hand, the liquid market diverts some of the illiquid asset's value as a risk-sharing instrument, pushing the OTC price down. The strength of these two effects depends on the risk profiles of the assets and the severity of the search friction. Evaluating the overall effect of the liquid market thus raises the following question: What is a meaningful combination of risk profiles and illiquidity level?

Looking at bonds and CDSs, or at mortgages and CDOs, or at any of the pairs listed above, we always observe the same pattern: The illiquid OTC market existed first and then the liquid market was created by financial intermediaries. The financial innovation is typically initiated either by an exchange or, if the new security trades OTC as well, by the dealers that will intermediate trades on the newly created market. In both cases, the intermediaries are interested in creating an active market.

Motivated by these examples, I put more structure into my model by endogenizing the risk-profile of the liquid asset. I assume that intermediaries select the liquid asset that maximizes the trading volume. Next, I compare the prices on the OTC market with and without the liquid asset. The risk profile of the optimal liquid asset is a weighted average of two risk profiles. The first profile is that of the illiquid asset, the second is the profile that would be optimal in terms of risk-sharing. I show that the weight on the profile of the illiquid asset is monotone increasing in the liquidity of the OTC market because, with a more active OTC market, there is more trading volume to capture. Perhaps paradoxically, this also means that the search frictions are easier to mitigate when they are smaller in the first place. Overall, the endogenous liquid asset increases the price on the OTC market when the search friction is sufficiently strong, but decreases it otherwise. Thus, the mitigation of the illiquidity discount dominates when the frictions on the OTC markets are strong enough, but the diversion of risk-sharing value away from the OTC market dominates otherwise.

This ambiguous equilibrium behavior is consistent with empirical findings regarding CDS trading and bond spreads. For example, Ashcraft and Santos [2009] find that the onset of CDS trading does not decrease the bond yield of the average firm, despite the new hedging opportunities. In my model, this corresponds to the illiquid market (the bond market) being at

understood as the returns on a CDS contracts for a protection seller. Indeed, holding a bond or selling protection on a CDS market offer essentially the same exposure to credit risk.

the threshold where the mitigation and diversion effects compensate each other. Differently, Saretto and Tookes [2013] find that CDS trading may have made debt financing less costly, but did so by relaxing only the "non-price" terms of debt.[6] In my model, this corresponds to the parameter range for which the mitigation effect dominates, meaning the range over which the trading frictions on the illiquid (bond) market are rather severe.

Finally, Das, Kalimipalli, and Nayak (2013) study empirically the interactions between CDS and bond markets and focus on informational efficiency. They provide "[...] *evidence of a likely demographic shift by large institutional traders from trading bonds to trading CDS in order to implement their credit views, resulting in declining efficiency and quality in bond markets.*" The same mechanism drives the findings in Das, Kalimipalli, and Nayak [2013] and the equilibrium behavior of my model. Specifically, Das et al. [2013] show how a CDS market diverts some trading away from the bond market, how this reduces the value of the bond, and how this reduction dominates any other benefits brought by the new market. This mechanism coincides with the equilibrium behavior of my model when the search frictions on the OTC market are not too severe.

Literature Review My paper builds on the literature considering the general equilibrium impact of trading frictions. These frictions can be the transaction costs on centralized markets, as in Lo, Mamaysky, and Wang [2004], Acharya and Pedersen [2005], Garleanu and Pedersen [2013], and Buss and Dumas [2013], or the search and bargaining frictions on OTC markets.[7]

The analysis of the search and bargaining frictions on OTC markets started, to a large extent, with Duffie et al. [2005]. Duffie et al. [2005] a model of OTC trading that shares features with job market models such as Diamond [1982].[8] The model that is most closely related to mine is Duffie, Gârleanu, and Pedersen [2007]. They also study bilateral trading in OTC markets with risk-averse agents. In comparison with Duffie et al. [2007], my main contribution is to model a second, liquid market and to study the interactions between the liquid and OTC markets. On the methodological side, I provide an existence and uniqueness argument that is also valid in Duffie et al. [2007]. In addition, I allow for more general aggregate shocks in the dynamic analysis of the model. Further references in asset pricing with search and bilateral trading include Weill [2008], Vayanos and Weill [2008], and Afonso and Lagos [2011]. A closely related strand of research considers centralized markets to which investors have intermittent and sometimes costly access. References in this strand include Lagos and Rocheteau [2007], Weill [2007], Lagos and Rocheteau [2009], and Gârleanu [2009]. In all these references, investors

[6]The *non-price* terms of a bond are its maturity, notional, and its many contract details (amortization, default triggers, embedded options, etc).

[7]Further trading restriction include portfolio constraints, margin requirements, and limited market participation. References in this literature include Merton [1987], Basak and Cuoco [1998], and Hugonnier [2012].

[8]See Mortensen [1987] or Rogerson, Shimer, and Wright [2005] for surveys of search models in labor economics.

trade on a unique market. My setting is also related to the one in Ang, Papanikolaou, and Westerfield [forthcoming], even if Ang et al. [forthcoming] focus on a porfolio problem in partial equilibrium. An extreme case of my model, the case in which trade on the OTC market is impossible, is related to the analysis in Longstaff [2009].

My paper also relates to the literature considering the interactions of different market structures. See, for example, Pagano [1989], Rust and Hall [2003], Miao [2006], and Vayanos and Wang [2007]. In these models, agents must execute a single trade and balance the benefits of a better price against a costly search. In contrast, in my model, investors trade repeatedly on both markets. Vayanos [1998] and Huang [2003] also analyze models in which agents trade assets with different liquidity levels. In both cases, however, the illiquidity is modeled by exogenous transaction costs. Exogenous and constant transaction costs cannot capture the endogenous and time varying interaction between demand shocks and illiquidity. Parlour and Winton [2013] discuss the role of monitoring when a bank chooses between selling loans and buying CDS protection. Biais [1993], Yin [2005], and De Frutos and Manzano [2002] statically compare prices on centralized and fragmented markets.

My paper is also related to the literature on hedging demand as a determinant of illiquidity discounts. In the context of bond markets, this mechanism is discussed, for instance, by Duffie [1996], Duffie and Singleton [1997], Krishnamurthy [2002], and Graveline and McBrady [2011].

My discussion of illiquidity spillovers is related to the literature that investigates contagion effects across markets. References focusing on volatility contagion include Hamao, Masulis, and Ng [1990], Lin, Engle, and Ito [1994], Kyle and Xiong [2001], Kodres and Pritsker [2002], and Hasler [2013]. References such as Chordia, Sarkar, and Subrahmanyam [2005] and Mancini, Ranaldo, and Wrampelmeyer [2013] document the cross-market effects of both returns and liquidity. My model explicitly describes a channel for illiquidity spillover effects.

Finally, the endogenous choice of the liquid asset in my model is similar to the security design setting in Duffie and Jackson [1989].

The outline of the paper is as follows. Section 1.2 introduces the model. Section 1.3 analyzes the investor's problem. Section 1.4 describes the population dynamics. Section 1.5 solves for an equilibrium. Section 1.6 considers the impact of aggregate demand shocks. Section 1.7 discusses the impact of the liquid market on the OTC market. Section 1.8 concludes.

1.2 Model

I study an economy in which investors share endowment risk by trading two different assets on, respectively, a liquid exchange and an OTC market. This model is an extension of Duffie, Gârleanu, and Pedersen [2007].

Assets and investors Two independent aggregate risk factors are described by the Brownian motions

$$\left(B_{a,t}, B_{b,t}\right)_{t\geq 0}.$$

Two risky assets, c and d, are exposed to these risk factors. The cumulative dividend payouts of these assets satisfy

$$\begin{aligned} \mathrm{d}D_{c,t} &= m_c \,\mathrm{d}t + a_c \,\mathrm{d}B_{a,t} + b_c \,\mathrm{d}B_{b,t}, \\ \mathrm{d}D_{d,t} &= m_d \,\mathrm{d}t + a_d \,\mathrm{d}B_{a,t} + b_d \,\mathrm{d}B_{b,t}. \end{aligned} \tag{1.1}$$

These assets are available in net supplies S_c and S_d, respectively. As described below, the asset c is traded on a *c*entralized market, whereas the asset d is traded on a *d*ecentralized, OTC market. For convenience, I define the vectors

$$e_c \triangleq \begin{pmatrix} a_c \\ b_c \end{pmatrix}, \, e_d \triangleq \begin{pmatrix} a_d \\ b_d \end{pmatrix}$$

and call them the exposures of the assets c and d, respectively.There is also a risk-free asset, available in perfectly elastic supply, and paying out dividends at the constant rate $r > 0$.[9]

The economy is populated by a continuum of investors. I write μ for a normalized measure over this continuum. Each investor receives an endowment driven both by the aggregate risk factors and by idiosyncratic shocks. More specifically, the cumulative endowment of investor i satisfies

$$\mathrm{d}\eta_t = m_\eta \,\mathrm{d}t + a_{i,t} \,\mathrm{d}B_{a,t} + b_{i,t} \,\mathrm{d}B_{b,t}, \tag{1.2}$$

and is thus driven by the two aggregate risk factors. The vector of exposures

$$e_{i,t} \triangleq \begin{pmatrix} a_{i,t} \\ b_{i,t} \end{pmatrix} \tag{1.3}$$

evolves stochastically. Specifically, the stochastic vector $e_{i,t}$ is a time-homogeneous Markov

[9] The interest rate is exogenous in all the models of asset pricing with search that I am aware of.

chain jumping back and forth between two (two-dimensional) values.[10] These two values are

$$e_1 \triangleq \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \text{ and } e_2 \triangleq \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad (\in \mathbb{R}^2)$$

and I denote by

$$\begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{pmatrix} \tag{1.4}$$

the generator of the Markov chain. The Markov chains are independent across agents.

There are various ways to interpret the idiosyncratic shocks to the vectors of exposures. A shock could represent a large loss incurred by an individual investing in assets that I do not model explicitly, significant inflows or outflows experienced by a fund, a significant movement in the inventory of a dealer, or the underwriting by a bank of a new bond issue. In all of these cases, an idiosyncratic shock calls for an adjustment of the risk exposure.[11] As summarized by Cochrane [2005], no matter what the exact interpretation of these shocks is, their role is to keep investors trading with each other.

Trading mechanisms Investors trade the liquid asset c on a centralized market. Investors access this market without delay and trade without transaction costs. The only minor restriction is that the number of shares held by an investor must belong to the range

$$[-K, K]$$

at any time, with $K > 0$ being a fixed, large number.[12] Investors also trade the risk-free asset at any time and without costs.

The other risky asset, d, is traded OTC. Trading d thus requires searching for a counter-party and negotiating the details of the transaction. The search process is governed by a “random matching". That is, a given investor gets in touch with another investor at the jump times of an idiosyncratic Poisson process with intensity Λ. This other investor is randomly drawn from across the population of investors. The draws are independent across meetings.

As the meeting intensity Λ controls the search friction on the OTC market, I call it the *liquidity*

[10] In particular, both components of the vector of exposures jump together.

[11] See Duffie et al. [2005] and Duffie et al. [2007] for discussions of these idiosyncratic preference shocks.

[12] This constraint is required by the verification argument et prevents doubling strategies. The constraint never binds in equilibrium when K is chosen large enough, as seen in Proposition 10 below. Gârleanu [2009] adopts the same type of restrictions.

of the OTC market. Given the dynamics of a Poisson process, the inverse

$$\xi \triangleq \frac{1}{\Lambda}$$

of the liquidity is the expected search time until the next meeting. I call this expected time the *illiquidity* of the OTC market.

Taking things together, investors from two separate subsets B and C of the population meet at the rate

$$2\Lambda\mu(B)\mu(C), \tag{1.5}$$

with μ being a measure on the set of investors. There is a factor 2 because the agents in B can both find an agent in C and be found by one.[13,14]

Once two agents have met, they bargain over a possible trade in the illiquid asset d. Specifically, they decide whether or not to exchange $\Theta > 0$ units of the asset and, if so, at which price. The outcome of the bargaining is given by the generalized Nash bargaining solution. Θ is an exogenous constant and the possible holdings in the illiquid asset d are restricted to two values, zero and Θ.[15]

Working with a continuum of agents makes the model tractable and isolates the effect of search frictions from, say, concerns of reputation or punishment. Moreover, actual OTC markets often involve large numbers of investors and dealers, making a detailed modeling of the entire market infeasible. For example, more than 600 dealers appear in the sample of corporate bond trades analyzed by Schultz [2001]. Similarly, Li and Schürhoff [2012] study a network of several hundred municipal bond dealers.[16]

[13] This statement is intuitive but non-trivial. More specifically, it assumes a certain Law of Large Numbers. See Duffie and Sun [2011] for the rigorous treatment of this issue in discrete time and Footnote 35 for a similar discussion.

[14] When appropriate, statements in this paper should be understood as holding almost surely.

[15] The assumption of a fixed transaction size is restrictive but both convenient and common. With fixed transaction size, the bargaining regarding the size of a transaction is reduced to accepting the trade or not. Alternatively, Lagos and Rocheteau [2007], Lagos and Rocheteau [2009], and Gârleanu [2009] let investors choose their holdings freely, but model an intermittent access to a centralized market instead of bilateral meetings. Afonso and Lagos [2011] and Cujean and Praz [2013] model bilateral markets with endogenous portfolio holdings.

[16] This hypothesis of a continuum of agents is common in the literature but sometimes criticized. The main criticism is as follows. Actual OTC market are often dominated by a limited number of dealers who form the core of the OTC market and market participants repeatedly and strategically interact with each other. Differently, in a model with a continuum of agents and random matching, investors never trade twice with the same person and this feature may prevent the model from capturing the working of actual OTC markets. This criticism is valid but the advantages of working with a continuum of investors are strong, and I decided to work with a continuum. In asset pricing with search, Duffie et al. [2005], Duffie et al. [2007], Vayanos and Wang [2007], and Weill [2007], for instance, model a continuum of investors who trade bilaterally. Alternatively, papers such as Gofman [2010], Gale and Kariv [2007], Malamud and Rostek [2012], and Babus and Kondor [2013] study decentralized trading among a

My second comment regards the matching technology (1.5). Other specifications exist in the literature.[17] However, the matching technology (1.5) has two advantages. First, as argued in Weill [2008], it results from an explicitly specified search process and the existence of this random matching is, partly, justified by the discrete time results in Duffie and Sun [2011]. Second, it exhibits increasing returns to scale. In the context of real assets, Gavazza [2011] argues that increasing returns to scale is an intuitively appealing and empirically important feature of search markets.

Preferences Each investor i maximizes her expected utility from consumption. Her utility function U has a constant coefficient of absolute risk aversion $\gamma > 0$ (exponential or CARA utility), meaning that

$$U : c \mapsto -e^{-\gamma x},$$

and their subjective rate of discounting is $\rho > 0$. The consumption and investment policy of i is thus dictated by the optimization

$$\sup_{\tilde{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(\tilde{c}_u)\,\mathrm{d}u \middle| \mathscr{F}_{i,0}\right], \tag{1.6}$$

with the admissible consumption processes $\tilde{c}$ satisfying certain conditions defined below and $\mathscr{F}_{i,0}$ being all the information available and relevant to i at time 0.

The payouts of the risky assets, defined in (1.1), are independent and identically distributed across time. Furthermore, the idiosyncratic exposure shocks defined by (1.4) offer a unique and stable stationary distribution of types 1 and 2 across the population. As a result, I expect all the aggregate quantities to be constant in the long run and I focus my analysis on this asymptotic, stationary case.[18,19] In a stationary equilibrium, the information set $\mathscr{F}_{i,0}$ only contains idiosyncratic quantities and the individual problem (1.6) becomes

$$V(w, i\theta) \triangleq \sup_{\tilde{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(\tilde{c}_u)\,\mathrm{d}u \middle| \begin{array}{rcl} w_0 & = & w \\ e_0 & = & e_i \\ \theta_0 & = & \theta \end{array}\right], \tag{1.7}$$

with w_0 being the wealth invested by i at time zero in either the liquid risky asset c or in the

finite number of investors.

[17] Weill [2008] and Inderst and Müller [2004], for instance, use more general matching technologies to study search problems in finance.

[18] I consider the impact of aggregate risk in Section 1.6.

[19] The cross-sectional distribution of wealth is not necessarily constant over time. However, the equilibrium policies are independent of the wealth, thanks to the CARA preferences, and this non-stationarity has no impact on the equilibrium portfolios and prices.

risk-free asset, e_0 being i's vector of exposures at time zero, and $\theta_0 \in \{0, \Theta\}$ being i's holdings in the illiquid asset at time zero. The optimization takes place over the set of consumption processes that satisfy the budget and admissibility constraints discussed below.

Budget constraint The consumption and trading of an investor must be consistent with the dynamics of her wealth. Specifically,

$$\mathrm{d}\tilde{w}_t = r\,\tilde{w}_t\,\mathrm{d}t - \tilde{c}_t\,\mathrm{d}t + \mathrm{d}\eta_t + \tilde{\theta}_t\,\mathrm{d}D_{dt} + \tilde{\pi}_t\,(\mathrm{d}D_{ct} - rP_c\,\mathrm{d}t) - P_d\,\mathrm{d}\tilde{\theta}_t, \tag{1.8}$$

with $\tilde{\pi}_t$ being the number of shares of the liquid asset c held at time t, P_c being the price of the liquid asset c and P_d being the price of the illiquid asset d. As the asset d is traded bilaterally, defining the dynamics of the holdings $\tilde{\theta}_t$ in d and the price P_d at which it trades requires some extra care.

Two investors trade the asset d if the transaction is in their best interest. Investor a sells the illiquid asset to investor b if there exists a price $\tilde{P}_d$ that satisfies both

$$V\left(w_a + \Theta\tilde{P}_d, i_a 0\right) \geq V\left(w_a, i_a, \Theta\right), \tag{1.9}$$

meaning that it is rational for a to sell, and

$$V\left(w_b - \Theta\tilde{P}_d, i_b \Theta\right) \geq V\left(w_b, i_b 0\right), \tag{1.10}$$

meaning that it is rational for b to buy.[20] If a trade is rational for the two counter-parties, the Nash bargaining solution determines the transaction price P_d. That is, P_d satisfies

$$P_d \in \arg\max_{\tilde{P}_d} \left\{ \begin{array}{c} \left(V\left(w_a + \Theta\tilde{P}_d, i_a, 0\right) - V\left(w_a, i_a, \Theta\right)\right)^{\eta_\Theta} \\ \cdot\left(V\left(w_b - \Theta\tilde{P}_d, i_b, \Theta\right) - V\left(w_b, i_b, 0\right)\right)^{\eta_0} \end{array} \right\}, \tag{1.11}$$

with $\eta_\Theta \in (0,1)$ being the bargaining power of the seller and $\eta_0 = 1 - \eta_\Theta$ being the bargaining power of the buyer.

The bilateral trading introduces a structure of rational expectations in the investor's problem. An investor takes as given both her own value function and those of the counter-parties she will meet. This investor then deduces the prices at which she will trade the illiquid asset, and deduces her own actual value function. A solution to the investor's problem thus consists in rational expectations regarding the value functions. I must still impose certain regularity conditions on the consumption processes.

[20]The wealth of a prior to the trade is w_a, its vector of exposures is e_{i_a}, etc.

Regularity The wealth process $\tilde{w}$ satisfies

$$\lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\tilde{w}_T}\right]=0. \tag{1.12}$$

The requirement (1.12) excludes pathological wealth processes and is needed in the verification argument for the Hamilton-Jacobi-Bellman (HJB) equations.[21] Pathological wealth processes include doubling strategies and the "financing" of consumption by an ever increasing amount of debt.

Finally, for ease of exposition, I restrict the model parameters as follows.

Assumption 1. The dynamic of the exposure shocks, as described by (1.4), and the supply S_d of the illiquid asset d satisfy

$$\frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}}\neq S_d\neq\frac{\lambda_{21}}{\lambda_{12}+\lambda_{21}}.$$

Also, the vectors e_d, e_c, and e_1-e_2 are not collinear. □

Assumption (1) prevents the lengthy treatment of non-generic cases.[22]

1.3 The Investor's Problem

I characterize the solution to the investor's problem (1.7) by the dynamic programming approach, meaning that I deduce the optimal consumption and trading policy from a HJB equation.[23]

Along an optimal path (π^*,θ^*,c^*,w^*), the process

$$\left(\int_0^t e^{-\rho s}U(c_s^*)\,\mathrm{d}s+e^{-\rho t}V\left(w^*,i_t\theta_t^*\right)\right)_{t\geq 0}=\left(\mathrm{E}\left[\int_0^\infty e^{-\rho s}U(c_s^*)\,\mathrm{d}s\,\middle|\,\mathscr{F}_t\right]\right)_{t\geq 0}$$

must be a martingale. Equating the expected dynamics of this process to zero and assuming that pointwise maximization characterizes the optimal consumption and investment policy

[21] The verification argument is the Appendix A.

[22] I endogenize the risk-profile e_c in Section 16. The assumption regarding the non-collinearity of the vectors of exposures will hold in this case if e_d and e_1-e_2 are not collinear

[23] The idiosyncratic exposure shocks and the idiosyncratic search processes make the markets incomplete. Further, the illiquid holdings can only be adjusted at stochastic times and are restricted to two values. A martingale approach along the lines of Karatzas, Lehoczky, and Shreve [1987] and Cox and Huang [1989] does not seem easily applicable.

yields the HJB equation

$$
\begin{aligned}
\rho V(w, i\theta) = \sup_{\tilde{c},\tilde{\pi}} \; & U(\tilde{c}) \\
& + \frac{\partial V}{\partial w}(w, i\theta)\left(r w - \tilde{c} + m_\eta + \theta m_d + \tilde{\pi}\,(m_c - r P_c)\right) \\
& + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(w, i\theta)\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}^* \\
& + \lambda_{i\bar{i}}\left(V(w, \bar{i}\theta) - V(w, i\theta)\right) \\
& + 2\Lambda\,\mathrm{E}^{\mu(b)}\left[\mathbf{1}_{\text{surplus}}\left(V(w - (\bar{\theta} - \theta)P_d, i\bar{\theta})) - V(w, i, \theta)\right)\right],
\end{aligned}
\tag{1.13}
$$

with the matrix of covariations

$$
\begin{aligned}
\Sigma_i &\triangleq \frac{1}{\mathrm{d}t}\,\mathrm{d}\left\langle \begin{pmatrix}\eta_t \\ D_{d,t} \\ D_{c,t}\end{pmatrix}, \begin{pmatrix}\eta_t & D_{d,t} & D_{c,t}\end{pmatrix}\right\rangle \\
&= \begin{pmatrix}
a_i^2 + b_i^2 & a_i a_d + b_i b_d & a_i a_c + b_i b_c \\
a_i a_d + b_i b_d & a_d^2 + b_d^2 & a_c a_d + b_c b_d \\
a_i a_c + b_i b_c & a_c a_d + b_c b_d & a_c^2 + b_c^2
\end{pmatrix},
\end{aligned}
\tag{1.14}
$$

for $i \in \{1,2\}$ and $\theta \in \{0, \Theta\}$.[24,25] On the right-hand side of (1.13), the fifth line represents the utility gains resulting from trading on the OTC market. The indicator function

$$\mathbf{1}_{\text{surplus}}$$

appears in the equation because not every meeting results in a trade. More specifically, two investors only exchange the illiquid asset if they have a surplus to share, meaning that both (1.9) and (1.10) hold. Furthermore, the transaction price P_d may depend on the counter-party b and is characterized by (1.11). The other terms on the right-hand side of (1.13) refer to the current consumption, the drift and volatility of the wealth, and shocks to the vector of exposures. To characterize, the solution to (1.13) I make the following assumption.[26]

Assumption 2. The value functions satisfy

$$V(w, i\theta) = -e^{-\alpha(w + a(i\theta) + \bar{a})},$$

for a set of numbers $\alpha \in \mathbb{R}_{>0}$, $a \in \mathbb{R}^4$, and $\bar{a} \in \mathbb{R}$ to be characterized. □

Assumption 2 will be justified *ex post* by an existence result. Conditional on Assumption 2, the

[24] I write "$\bar{\cdot}$" for "the other possible value". For example, if $i = 1$, then $\bar{i} = 2$.

[25] For convenience, I index the entries of Σ_i by i, d, and c.

[26] This functional form is standard for problems similar to the one at hand. It is used, among others, by Duffie, Gârleanu, and Pedersen [2007], Vayanos and Weill [2008], and Gârleanu [2009].

optimal policy of the investors is known in closed-form. First, the trading on the OTC market occurs as described below.

Proposition 3 (OTC trading)**.** *On the OTC market, investors trade as follows.*

1. *Investors with exposure type 1 sell the illiquid asset to those with type 2 exposure when*

$$a(2\Theta) - a(20) > a(1\Theta) - a(10). \tag{1.15}$$

   In particular, the decision to trade does not depend on the wealth of the investors.

2. *Investors with exposure type 2 sell the illiquid asset to those with type 1 exposure when*

$$a(2\Theta) - a(20) < a(1\Theta) - a(10). \tag{1.16}$$

   The decision to trade does not depend on the wealth of the investors either.

3. *If a i-investor sells the illiquid asset to a $\bar{i}$-investor, the transaction price P_d is the unique solution to*

$$\left(1-\eta_0\right)\left(1-e^{\alpha(a(i0)+P_d\Theta-a(i\Theta))}\right) = \eta_0\left(1-e^{\alpha\left(a(\bar{i}\Theta)-P_d\Theta-a(\bar{i}0)\right)}\right). \tag{1.17}$$

   This solution is available in closed-form.[27]

Proof. See Proof 38 in Appendix D. □

The proposition shows that the bargaining outcomes on the OTC market depend on the exposures (and holdings) of the counter-parties but not on their wealth. As investors are only interested in the rest of the population to the extent that it represents potential counter-parties, I call *type* of an agent the combination of her exposure, indexed by 1 or 2, and her illiquid holdings, 0 or Θ.

The characterization of the transactions on the OTC market in terms of inequalities (1.15) and (1.16) is intuitive. Recalling Assumption 2 regarding the value functions, the difference

$$v(i\Theta) - v(i0), \ i = 1, 2,$$

[27] Equation (1.17) is quadratic in

$$x \stackrel{\Delta}{=} \exp\left(\alpha\Theta P_d\right) \ (>0)$$

and admits a unique positive solution. This unique solution readily defines P_d.

is the reservation value of a i-agent for the illiquid asset. Inequality (1.15) thus states that the reservation value of the 2-agents is higher than that of the 1-agents. In this case, the 2-investors buy the illiquid asset from the 1-investors. Inequality (1.16) is just the opposite.

I can also characterize the optimal consumption and investment in the liquid asset.

Proposition 4 (Consumption and Liquid Holdings)**.** *The optimal consumption is*

$$c(i\theta) = \frac{1}{\gamma}\left(\alpha\left(w + a(i\theta) + \bar{a}\right) - \log\left(\frac{\alpha}{\gamma}\right)\right)$$

and the optimal holdings in the liquid asset are

$$\pi(i\theta) = \frac{1}{\Sigma_{cc}}\left(\frac{1}{\alpha}\left(m_c - rP_c\right) - \left(\Sigma_{ic} + \Sigma_{cd}\right)\right), \tag{1.18}$$

for any type $i\theta$ and liquid wealth w.

Proof. See Proof 39 in Appendix D. □

Combining the HJB equation (1.13) with Lemma 3 and Lemma 4 provides a narrower characterization of the value function.

Proposition 5 (Value Functions)**.** *The constants in the value function*

$$V(w, i\theta) = -\exp\left(-\alpha\left(w + a(i\theta) + \bar{a}\right)\right),$$

are characterized as follows. First, $\alpha = r\gamma$. Furthermore, choosing

$$\bar{a} = \frac{1}{r\gamma}\left(-1 + \frac{\rho}{r} + \gamma m_\eta + \log(r)\right), \tag{1.19}$$

and taking the cross-sectional distribution of types $\mu \overset{(\Delta)}{=} \{\mu(i\theta)\}_{i\theta}$ as given, the type specific constants "$a(i\theta)$" are the unique solution to the system

$$ra(i\theta) = \kappa(i\theta) + \lambda_{i\bar{i}}\left(\frac{e^{-r\gamma(a(\bar{i}\theta) - a(i\theta))} - 1}{-r\gamma}\right) + 2\Lambda\mu(\bar{i}\bar{\theta})\left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}(a)\right)}{-r\gamma}\right]^+, \qquad i \in \{1,2\}, \theta \in \{0,\Theta\}, \tag{1.20}$$

with the quantity

$$\kappa(i\theta) \overset{\Delta}{=} \theta m_d + \pi(i\theta)\left(m_c - rP_c\right) - \frac{1}{2}r\gamma\begin{pmatrix}1 & \theta & \pi(i\theta)\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta & \pi(i\theta)\end{pmatrix}^*, \tag{1.21}$$

measuring the mean-variance benefits of the risk-profile, the function

$$\epsilon_{i\theta} : \{a\}_{i\theta} \mapsto a(\bar{i}\theta) - a(\bar{i}\bar{\theta}) + a(i\bar{\theta}) - a(i\theta) \tag{1.22}$$

measuring the surplus that a $i\theta$-investor may be able to share on the OTC market, and the function

$$\begin{aligned} \chi : (0,1) \times \mathbb{R} &\to \mathbb{R} \\ (\eta, \epsilon) &\mapsto \frac{2\left(1-\eta\right)}{1 - 2\eta + \sqrt{(2\eta-1)^2 + 4\eta(1-\eta)e^{r\gamma\epsilon}}} - 1 \end{aligned} \tag{1.23}$$

mapping a bargaining power and a surplus to the utility change induced by a trade OTC.[28,29,30]

Proof. See Proof 40 in Appendix D. □

Equation (1.20) decomposes the expected utility of an agent into the sum of three terms. First, there is a flow of mean-variance benefits resulting from an investor's risk profile. Second, there are the shocks to the vector of exposures. Finally, there are the benefits resulting from trading on the OTC market. The benefits resulting from trading on the liquid market do not appear explicitly in Equation (1.20) but are contained in the flow "$\kappa(i\theta)$" of mean-variance benefits.

Intuitively, the quantity

$$\mathscr{S} \stackrel{\Delta}{=} (\kappa(2\Theta) - \kappa(20)) + (\kappa(10) - \kappa(1\Theta)) \tag{1.24}$$

indicates whether transferring the illiquid asset from a 1Θ-investor to a 20-investor increases the overall flow of mean-variance benefits.[31] $\mathscr{S}$ should then also indicate whether a sale of the illiquid asset by a 1Θ-investor to a 20-investor is profitable in equilibrium or not. As stated in Proposition 10 below, this is indeed the case. To characterize an equilibrium, I must first characterize the distribution of types across the population.

[28] I abuse notation and write μ both for a measure on the set of investors and for the distribution of the type of the investors under μ. μ defines the type distribution but not the other way around.

[29] I write $[x]^+ \stackrel{\Delta}{=} \max\{0, x\}$ for the positive part of a number.

[30] When all investors have the same bargaining powers, $\eta_\theta = 1/2$, the function χ significantly simplifies. Namely,

$$\chi\left(\frac{1}{2}, x\right) = \frac{1}{\sqrt{e^x}} - 1.$$

[31] I write $\mathscr{S}$ like in risk *s*haring.

1.4 Cross-Sectional Distribution of Types

The type of a given agent changes either because of a shock in her endowment correlations, or because she traded on the decentralized market. As described in Proposition 3, two mutually exclusive trade patterns can exist on the decentralized market, depending on which agents have the higher valuation. Investors endogenously choose which trading pattern they follow but, for this section, I assume the following.

Assumption 6. Agents with the exposure type 2 buy the illiquid asset.

Recalling both the dynamics of the endowment correlations assumed in Equation 1.4 and the linear matching technology assumed in Equation 1.5, the type distribution μ must satisfy the stationary Kolmogorov forward equation

$$\begin{cases} 0 = \dot{\mu}(10) & = & 2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{12}\mu(1l) & +\lambda_{21}\mu(2l) \\ 0 = \dot{\mu}(1\Theta) & = & -2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{12}\mu(1h) & +\lambda_{21}\mu(2\Theta) \\ 0 = \dot{\mu}(2l) & = & -2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{21}\mu(2l) & +\lambda_{12}\mu(1l) \\ 0 = \dot{\mu}(2\Theta) & = & 2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{21}\mu(2\Theta) & +\lambda_{12}\mu(1\Theta) \end{cases} . \tag{1.25}$$

On the right-hand side of each equation, the first term refers to trading, and the other ones to endowment shocks.[32] Also, μ, being a density, must satisfy both

$$\mu(10) + \mu(1\Theta) + \mu(20) + \mu(2\Theta) = 1 \tag{1.26}$$

and

$$\big(\mu(1\Theta), \mu(10), \mu(2\Theta), \mu(20)\big) \in \mathbb{R}^4_+. \tag{1.27}$$

Finally, the OTC market has to clear, meaning that every unit of the illiquid asset d must be held by someone. This is expressed by imposing the condition

$$\Theta\big(\mu(1\Theta) + \mu(2\Theta)\big) = S_d. \tag{1.28}$$

As seen from (1.26), (1.27), and (1.28), the population can absorb at most Θ units of the asset d. This imposes the constraint

$$0 \leq \frac{S_d}{\Theta} \leq 1 \tag{1.29}$$

on the exogenous parameters of the model.

[32] The terms referring to trading only involves trades between agents with different endowment correlation. However, according to Proposition 3, agents with the same endowment correlations, but different holdings will also trade. However, as such agents will only swap their types, this has no impact on the distribution of types.

As shown in Duffie et al. [2005], the system defining the stationary distribution is well-behaved. I recall their result for convenience.

Proposition 7 (Duffie et al. [2005], Proposition 1)**.** *There exists a unique stationary type distribution that is reached from any initial distribution.*

Proof. For convenience, I partially recall the argument of Duffie et al. [2005] in Proof 41, in Appendix D. □

If Assumption 6 fails and 1-investors buy the illiquid asset, then all statements in this section 1.4 are still valid, up to a systematic swap of the indexes 1 and 2.[33]

There are thus only two possible stationary distribution. I denote the one arising under Assumption 6 by $\mu^{1h\to 2l}$ and the other by $\mu^{2h\to 1l}$. In equilibrium, the trade surplus

$$\epsilon_{i\theta} \overset{(\Delta)}{=} a(i\bar{\theta}) - a(i\theta) + a(\bar{i}\theta) - a(\bar{i}\bar{\theta})$$

decides which type distribution obtains, in the sense that

$$\mu(a) = \mathbf{1}_{\{\epsilon_{1\Theta}(a)>0\}}\mu^{1\Theta\to 2l} + \mathbf{1}_{\{\epsilon_{2\Theta}(a)>0\}}\mu^{2\Theta\to 1l}. \tag{1.30}$$

The trade surpluses define the trading pattern on the OTC market via the Nash bargaining solution characterized in Proposition 3. In turn, the trading pattern defines the stationary distribution of types via the flow equations (1.25).

Finally, it may be useful to consider the behavior of the type distribution when the OTC market is relatively liquid. The exact asymptotic behavior depends on the relationship between the supply S_d of the illiquid asset and the proportion of investors

$$\mu_2 \overset{\Delta}{=} \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}}$$

[33] Strictly speaking, I should still consider the borderline case for which all investors have the same reservation value for the asset and are indifferent between buying and selling the illiquid asset. This case can arise but is non-generic in the exogenous parameters. Further, when the surplus to share on the OTC market is exactly zero, I should make additional assumptions regarding when a bilateral trade occurs. For these two reasons, I do not explicitly analyze this case. In proposition 10 I exactly characterize when this non-generic case arises.

having a high valuation for this illiquid asset.[34,35] In remaining of the paper I assume

$$\mu_2 > \frac{S_d}{\Theta}. \tag{1.31}$$

Under this assumption, the "marginal" buyer of the asset d in a Walrasian setting would have a high valuation for the asset. By *marginal buyer* I mean the investor that would buy the additional units of the illiquid asset, should S_d be marginally increased.[36]

Proposition 8. *Under assumption* (1.31)*, the equilibrium density satisfies*

$$\begin{pmatrix} \mu(10) \\ \mu(1\Theta) \\ \mu(20) \\ \mu(2\Theta) \end{pmatrix} = \mu^W + \frac{1}{\Lambda}\delta_\mu \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + o\left(\frac{1}{\lambda}\right)\mathbf{1}_4, \tag{1.32}$$

with the limit value and sensitivities

$$\mu^W \triangleq \begin{pmatrix} 1 - \frac{S_d}{\Theta} \\ \frac{S_d}{\Theta} - \mu_2 \\ 0 \\ \mu_{2\bullet} \end{pmatrix}, \qquad \delta_\mu \triangleq \frac{\lambda_{12}}{2} \frac{\frac{S_d}{\Theta}}{\mu_2 - \frac{S_d}{\Theta}}$$

and with $\mathbf{1}_4 \in \mathbb{R}^4$ being the vector whose components are all equal to 1.

Proof. See Proof 44 in Appendix D. □

The asymptotic expressions above can be understood intuitively. The common absolute value of the four components of the first order correction reflects the functioning of the

[34] Similarly, I define

$$\mu_1 \triangleq \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}}$$

as the proportion of investors with a low valuation for the illiquid asset.

[35] Recalling (1.4), the description of μ_2 as the proportion of 2-investors across the population intuitive. Whether it is correct or not, however, is far from trivial. Indeed, this statement identifies the proportion of time spent by a given agent in a given state and the proportion of agents across the population who currently are in that state. This requires the application of a certain Law of Large Numbers across the population. See Sun [2006] for a rigorous treatment of this issue.

[36] All the derivations could be done assuming the inequality opposite to 1.31. However, assumption 1.31 makes sure that the illiquidity discount is positive, meaning that the illiquidity of the OTC market decreases the value of the asset OTC. Thinking about bond markets and the well-documented positive liquidity spreads on bonds, this seems to be a desirable model feature. For this reason, presumably, assumption (1.31) and variants thereof recurrently appear in the literature. See, for instance, Condition 1 in Duffie et al. [2005], Equation (1) in Weill [2008], or Assumption 2 in Vayanos and Weill [2008].

decentralized market. That is, every time one potential buyer and one potential seller meet, a transaction occurs, and they become a satisfied holder of the asset d and one satisfied non-holder, respectively.

1.5 Stationary Equilibrium

In this section I define and characterize an equilibrium of the model. In other words, I make the individual decisions consistent with the aggregate quantities in the economy.

For the centralized market, I use a classical Walrasian equilibrium concept. Specifically, as seen in Proposition 4, the only aggregate quantity affecting the liquid holdings is the price P_c of the liquid asset. I thus impose the consistency between the individual and aggregate quantities by requiring the price P_c to be so that the centralized market clears.

Turning to the OTC market, the decisions to trade or not and, if so, at which price, are dictated by the parametrization

$$a = \{a(i\theta)\}_{i\theta}$$

of the value functions (see Proposition 3).

Now, on the one hand, the individual trading decisions on the OTC market yield a certain type distribution, characterized in Section 1.4. On the other hand, a also depends on the distribution of types across the population. This is clear at both the intuitive and technical levels. Intuitively, it is clear because the utility of an investor searching for a counterparty on an OTC market should depend on the likelihood of finding such a counterparty. Technically, it is clear because a is a solution to the HJB equation (1.20), an equation in which the distribution μ appears.

I will thus impose the equilibrium condition that the type distribution assumed when writing the HJB equation (1.20) and the one generated by the solution to (1.20) are equal. I formalize this discussion as follows.

Definition 9. A *stationary equilibrium* of the model consists of a price P_c $(\in \mathbb{R})$, a collection of liquid holdings $\{\pi(i\theta)\}_{i\theta}$ $\left(\in \mathbb{R}^4\right)$ corresponding to each type, a distribution of types $\{\mu(i\theta)\}_{i\theta}$ $\left(\in \mathbb{R}^4\right)$, and the constants $\{a(i\theta)\}_{i\theta}$ $\left(\in \mathbb{R}^4\right)$ defining the value functions. The equilibrium quantities satisfy three conditions.

1. An investor of type $i\theta$ who takes the price P_c as given optimally invests the amount $\{\pi(i\theta)\}_{i\theta}$ in the liquid asset c.

2. The centralized market clears, meaning that

$$\mathrm{E}^{\mu(i\theta)}\left[\pi(i\theta)\right] = S_c.$$

3. The value functions and stationary type distribution are consistent. Specifically, the vector

$$a = \{a(i\theta)\}_{i\theta}$$

solves the HJB equation (1.20) when the type distribution is $\mu\left(\epsilon_{1h}(a)\right)$.[37]

I now state the main result of this section.

Proposition 10. *There exists exactly one equilibrium of the model. In equilibrium, 2-agents have a high valuation of the illiquid asset d exactly when*

$$\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right) > 0, \tag{1.33}$$

with e_c, e_d, e_1, and e_2 being the exposures to the aggregate factors of the liquid asset, the illiquid asset, and the endowments, respectively.

Furthermore, the equilibrium price of the liquid asset is

$$P_c = \frac{m_c}{r} - \gamma\left(\Sigma_{cc} S_c + \Sigma_{\eta c} + \Sigma_{cd} S_d\right), \tag{1.34}$$

with

$$\Sigma_{\eta c} \triangleq \mu_1 \Sigma_{1c} + \mu_2 \Sigma_{2c}$$

being the average correlation between the endowments and the dividends of the liquid asset, and the equilibrium holdings of the four types are

$$\pi(i\theta) = S_c + \frac{1}{\Sigma_{cc}}\left(\left(\Sigma_{\eta c} - \Sigma_{ic}\right) + \Sigma_{cd}\left(S_d - \theta\right)\right), \tag{1.35}$$

for $i = \{1, 2\}$ and $\theta = \{0, \Theta\}$.

Proof. See Proof 45 in Appendix D. □

As Equation (1.34) shows, the liquidity Λ of the OTC market does not affect the price of the liquid asset. This result can be understood intuitively. Indeed, the search friction makes the

[37] The distribution $\mu\left(\epsilon_{1h}(a)\right)$, a function of the trade surplus, is defined in (1.30).

asset allocation inefficient. Illiquidity thus increases the proportion of investors who short the liquid asset because they have not yet found a counter-party to buy their illiquid holdings. These investors reduces the aggregate demand for the liquid asset. At the same time, the search friction also increases the number of investors who buy the liquid asset while trying to increase their illiquid holdings. This second demand compensates the first one and, overall, the search friction does not affect the price of the liquid asset. It is true, however, that the two types of hedging demand exactly offset each other because there is no aggregate uncertainty. In Section 1.6, I consider aggregate demand shocks and, in this case, the frictions in the OTC market affect the price of the liquid asset.

Having a market friction that impacts individual policies but not prices is reminiscent of several references. Gârleanu and Pedersen [2004] documents a similar effect in a setting with adverse selection, and so does Gârleanu [2009] in a setting with search friction and a unique market. The conclusions of Rostek and Weretka [2011] are similar as well, but with illiquidity measured as a price impact.

There is also a short technical argument explaining why the equilibrium price of the liquid asset is independent of the illiquidity of the OTC market. Proposition 4 shows that the optimal holdings in the liquid asset are linear both in the covariance Σ_{ic} of the endowment with the payouts of c and in the illiquid holdings θ. Now, the cross-sectional average of the endowment correlations is independent of the illiquidity of the decentralized market. Indeed, the correlations define which holdings agents intend to hold, and this is independent of how much time it will actually take to obtain these holdings. Similarly, the cross-sectional average of the illiquid holdings is a matter a market-clearing, and not of illiquidity per se. Taking things together, as the optimal holdings in the illiquid asset are linear in the model parameters, and as the cross-sectional averages of these parameters are independent of the liquidity level, so is the aggregate demand, and so is the price of the liquid asset c.

The condition (1.33) characterizes which investors have the higher valuation for the illiquid asset and is rather intuitive. The first term of the product,

$$\det\left(\left(\begin{array}{c|c} e_d & e_c \end{array}\right)\right),$$

measures how orthogonal the risk profiles of the liquid and illiquid assets are. Phrased differently, this first term measures how much risk sharing can be achieved on the OTC market only. The second term in the product,

$$\det\left(\left(\begin{array}{c|c} e_1 - e_2 & e_c \end{array}\right)\right),$$

again compares how orthogonal two vector of exposures are. The first vector, $e_1 - e_2$, is the risk-profile that 2-investors should buy to achieve an optimal risk-sharing. The second vector is again the risk profile of the liquid asset. As a result, this second term measures how much risk-sharing is left once agents have chosen their exposure on the liquid market.

Finally, the product of the two terms is positive exactly if the exposure that is specific to the OTC market and the risk-sharing that cannot be achieved on the liquid market "overlap". Figure 1.2 offers a visual interpretation of the condition (1.33).

This discussion shows that the buyers of the illiquid asset are necessarily those investors to whom the fundamentals of the illiquid asset offer high diversification benefits. In particular, even if illiquidity distorts the value functions and prices, it does not modify an agent's decision to hold an asset or not. Even in an illiquid market, the fundamentals of the asset guide this decision.

Finally, the condition (1.33) is equivalent to $\mathscr{S} > 0$, with $\mathscr{S}$ defined in (1.24).

In general, the equilibrium quantities for the decentralized market are cumbersome to deal with. Specifically, I cannot characterize the parametrization a of the value functions in closed-form. Explicit expressions for a certain asymptotic case are available in Appendix A.

On the technical side, the existence and uniqueness result of Proposition 10 appears to be new. More specifically, in Duffie, Gârleanu, and Pedersen [2005], the equilibrium quantities are known in closed-form but only because agents are assumed to be risk neutral. This setting was then extended by Duffie, Gârleanu, and Pedersen [2007], Vayanos and Weill [2008], and others to accommodate risk-averse (CARA) agents. In these cases, the authors showed how, asymptotically, the solutions to these models were formally equivalent to those encountered in settings with risk-neutral agents. However, the asymptotic analyses involved either a vanishing risk-aversion, or a vanishing heterogeneity of the agents. My argument does not need these assumptions.[38]

Thanks to Proposition 10, I can characterize the equilibrium dispersion of holdings in the liquid asset.

Corollary 11. *The mean absolute deviation of the holdings in c,*

$$\mathrm{E}^{\mu(i\theta)}\left[|\pi(i\theta) - S_c|\right],$$

is increasing in the illiquidity $\frac{1}{\Lambda}$ of the OTC market.

The proof of Corollary 11 follows directly from (the proof) of Proposition 7 and from Proposition 10. Proposition 7 characterizes the equilibrium distribution of types and Proposition 10

[38] Gârleanu [2009] sketches an existence argument for an alternative model of illiquid market. However, in his setting, prices are Walrasian and not bargained, which modifies the structure of the equilibrium.

characterizes the equilibrium holdings in c.

Corollary 11 is exactly in line with the findings of Oehmke and Zawadowski [2013] regarding CDS and bond markets. Indeed, Oehmke and Zawadowski [2013] find that the CDS market is used as a liquid alternative to an illiquid bond market, and that the dispersion of holdings in CDS contracts is increasing in the illiquidity of the bond market.

My model also has predictions regarding the trading volumes on the two markets. I define the trading volume as the number of shares of an asset that is traded per unit of time. On the illiquid market, trades occur at the rate

$$2\Lambda\mu(1\Theta)\mu(20)$$

when the inequality (1.33) holds, and each trade involves the exchange of Θ units of the illiquid asset. The trading volume on the illiquid market is thus

$$2\Lambda\mu(1\Theta)\mu(20)\Theta.$$

Investors trade infrequently even on the liquid market. This is a consequence of the CARA preferences and of the trade motives being driven by infrequent jumps and not by diffusions. Namely, there are six possible type changes and each of them occurs with a given intensity. These six type changes are

type change	rate	triggered by
$10 \to 20$	$\lambda_{12}\mu(10)$	correlation shock
$1\Theta \to 2\Theta$	$\lambda_{12}\mu(1\Theta)$	correlation shock
$1\Theta \to 10$	$2\Lambda\mu(1\Theta)\mu(20)$	OTC trade
$20 \to 2\Theta$	$2\Lambda\mu(1\Theta)\mu(20)$	OTC trade
$20 \to 10$	$\lambda_{21}\mu(20)$	correlation shock
$2\Theta \to 1\Theta$	$\lambda_{21}\mu(2\Theta)$	correlation shock

.

The trading volume on the liquid market is then

$$\begin{aligned}
\mathrm{Vol} &= \frac{1}{2}\left(\sum_{\text{type changes}} (\text{rate}) \times (\text{size of trade})\right) \\
&= \frac{1}{2}\left\{\begin{array}{rl}
\lambda_{12}\mu(10) & |\pi(10)-\pi(20)| \\
+\lambda_{12}\mu(1\Theta) & |\pi(1\Theta)-\pi(2\Theta)| \\
+2\Lambda\mu(1\Theta)\mu(20) & |\pi(1\Theta)-\pi(10)| \\
+2\Lambda\mu(1\Theta)\mu(20) & |\pi(20)-\pi(2\Theta)| \\
+\lambda_{21}\mu(20) & |\pi(20)-\pi(10)| \\
+\lambda_{21}\mu(2\Theta) & |\pi(2\Theta)-\pi(1\Theta)|
\end{array}\right\},
\end{aligned}$$

with the factor 1/2 recalling that each purchase must be matched by a sale.

Combining (the proofs of) Propositions 7 and Proposition 10 yields the following:

Corollary 12. *The trading volumes on* both *markets are decreasing in the illiquidity level* $\xi = 1/\Lambda$.

Proof. See Proof 46 in Appendix D. □

Corollary 12 shows how the liquid and the illiquid assets are complements in terms of trading volumes, with the trading volumes on both markets increasing and decreasing together. Interestingly, this relationship between the trading volumes holds independently of whether the assets are complements or substitutes in terms of risk-exposure. This relationship between the trading volumes is driven by the use of the liquid asset as a hedging instrument, as detailed in Proposition 13.

Proposition 12 also deserves to be compared with a result from Longstaff [2009]. Longstaff [2009] proposes a model in which, for a given period, only one of two assets can be traded. Longstaff [2009] then concludes that illiquidity increases a certain measure of the trading activity in the liquid asset. This is obviously in contradiction with my conclusion. The origin of this divergence is in the preferences of the agents.

In my model, investors with CARA preferences intend to keep their holdings fixed essentially all the time. A re-balancing is triggered only by a preference shock or by a wish to adjust the liquid holdings as the result of a change in the illiquid holdings. However, making bilateral transactions more difficult makes the trade motives even less frequent, and reduces trading volumes.

Quite differently, in Longstaff [2009], investors have a constant relative risk aversion (CRRA) and want to constantly re-balance their holdings in both assets. Now, if trading in one of them is impeded, this is compensated by trading the other one more intensively, which induces the trading volume increase.

I now provide a more detailed characterization of trading on the centralized market.

Proposition 13. *Opening the OTC market strictly increases the trading volume Vol on the liquid market.*

Furthermore, if the inequality

$$\det\left(\left(\begin{array}{c|c} e_d & e_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c|c} e_1 - e_2 & e_c \end{array}\right)\right) (e_c \cdot (e_1 - e_2))\,(e_c \cdot e_d) > 0 \tag{1.36}$$

holds, the search frictions on the OTC market discontinuously increase the trading volume on the liquid market. In mathematical terms,

$$\lim_{\Lambda\to\infty} Vol(\Lambda) > Vol_W,$$

with Vol_W being the trading volume in c if the asset d is traded on a competitive (Walrasian) market.[39] *If* (1.36) *does not hold, the asymptotic trading volume with a vanishing friction and the Walrasian trading volume coincide.*

Proof. See Proof 47 in Appendix D. □

This last result shows how the search friction on the OTC market can generate some additional, or "excessive", trading on the centralized market. Intuitively, this arises from the investors' use of the liquid asset as an imperfect substitute for the illiquid asset.

1.6 Aggregate Demand Shocks

The price impact of illiquidity can be driven both by the illiquidity level and by the illiquidity risk, that is, by time variation in illiquidity.[40]

In my model, liquidity is understood as the time it takes to complete a transaction on the OTC market. More specifically, in the steady state, an investor who is attempting to sell her illiquid holdings measures illiquidity as

$$\frac{1}{\Lambda}\frac{1}{\mathrm{P}\left[\text{sell the asset}\,|\,\text{contacted an investor}\right]},$$

meaning as the expected time until the sale is completed. The meeting intensity Λ represent the technology used by investors to contact each others and is unlikely to change in unpredictable ways over time. The probability of completing the trade, $\mathrm{P}\left[\text{sell}|\,\text{contacted}\right]$, however, is determined by the distribution of preferences across the population of investors and this distribution can reasonably be assumed to evolve stochastically, leading to illiquidity risk.

In this section, I introduce time variation in liquidity by assuming that the proportion of agents with a high valuation for the illiquid asset is driven both by aggregate and by idiosyncratic

[39] I consider a Walrasian setting in which investors can trade d whenever they want, at no cost, and taking the price $P_{d,W}$ of the asset d as given, but I maintain the constraint that the holdings in d must belong to $\{0,\Theta\}$.

[40] The importance of illiquidity risk was emphasized by Pastor and Stambaugh [2003] and Acharya and Pedersen [2005], and further analyzed by Bongaerts et al. [2011], Junge and Trolle [2013], and Mancini et al. [2013].

shocks. The aggregate shocks occur at the jump times of the Poisson process

$$\left(N_t^a\right)_{t\geq 0}$$

whose intensity is λ_a. The proportion of 2-agents after such an aggregate shock is drawn from a distribution M_2 and the draws are independent across aggregate shocks.

I still assume *ex ante* that 2-agents have the high valuation and verify *ex post* this assumption. Furthermore, I assume that the support of the distribution after an aggregate shock satisfies

$$\operatorname{supp}(M_2) \subset \left(\frac{S_d}{\Theta}, 1\right].$$

This maintains a high valuation for marginal investors at any time or, equivalently, maintains a positive illiquidity discount.

In mathematical terms, these assumptions translate into the dynamics

$$\mathrm{d}\mu_2(t) = -\lambda_{21}\mu_2(t-) + \lambda_{12}\mu_1(t-) + \big(m_2 - \mu_2(t-)\big)\,\mathrm{d}N_t^a,\ m_2 \sim M_2, \tag{1.37}$$

for the proportion $\mu_2(t)$ of 2-investors at time t. Note that, between two aggregate shocks, this proportion evolves deterministically. Using a formalism similar to the one in Duffie et al. [2005], I index the state of the system by the last aggregate shock and the time elapsed since this last shock.[41,42] For example, the proportion of 2-investors t units of time after it jumped to m_2 is

$$\mu_2(m_2, t) = e^{-(\lambda_{12}+\lambda_{21})t} m_2 + \left(1 - e^{-(\lambda_{12}+\lambda_{21})t}\right)\frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}} \tag{1.38}$$

In particular, if no aggregate shock has occurred for a long time, and independently of the last shock, the proportion of 2-investors converges toward the same level. I write

$$\mu_2(\infty) \triangleq \frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}}$$

[41] In Duffie et al. [2005] the distribution after the shock is concentrated on one point, meaning that the state of the system can be indexed by the time since the last shock only.

[42] The last aggregate shock should be represented by the entire distribution

$$\left(\mu^a(1l), \mu^a(1h), \mu^a(2l), \mu^a(2h)\right)$$

reached after the aggregate exposure shock occurred. In the asymptotic case I consider, however, only the proportion

$$\mu_2^a \overset{(\Delta)}{=} \mu^a(2l) + \mu^a(2h)$$

of high valuation investors matter. As a result, I abuse notations and index the current state of the economy by the last draw from m_a

for this level.[43]

I must still specify the shocks at the individual level that will generate the aggregate dynamics (1.37). I do so by assuming, for a proportion of 2-investors jumping from $\mu_2(t-)$ to m_2, that each 2-investor has a probability

$$\delta\left(2;\mu_2(t-);m_2\right) \triangleq \max\left\{0;\frac{\mu_2(t-)-m_2}{\mu_2(t-)}\right\} \tag{1.39}$$

of becoming type 1 and that each 1-investor has a probability

$$\delta(1;\mu_2(t-);m_2) \triangleq \max\left\{0;\frac{m_2-\mu_2(t-)}{1-\mu_2(t-)}\right\} \tag{1.40}$$

of becoming type 2. Assuming that a suitable version of the strong law of large numbers (SLLN) holds cross-sectionally, these idiosyncratic shocks will be consistent with the aggregate dynamics (1.37).[44]

As far as the type distribution is concerned, the state of the economy can be described by the last aggregate valuation shock and the time elapsed since that shock. Furthermore, the type distribution evolves continuously between two aggregate valuation shocks. I thus assume the same type of evolution for both the price of the liquid asset and the value functions.

I assume that the equilibrium price process of the liquid asset is a function

$$P_c\left(m_2,t\right) \tag{1.41}$$

of the last aggregate liquidity shock and of the time elapsed since this last shock. I also assume that this price is differentiable in time. Under this assumption on the price process, the budget constraint of an investor becomes

$$\mathrm{d}w_t = r\,w_t\,\mathrm{d}t - c_t\,\mathrm{d}t + \mathrm{d}e_t + \theta_t\,\mathrm{d}D_{d,t} + \pi_t\left(\dot{P}_{c,t}\,\mathrm{d}t + \mathrm{d}D_{c,t} - rP_{c,t}\right) - P_{d,t}\,\mathrm{d}\theta_t, \tag{1.42}$$

with w being the wealth invested at the risk-free rate or in the liquid asset, c being the consumption, θ being the holdings in the illiquid asset, π being the holdings in the liquid asset,

[43] In the stationary setting of Section 1.5, I simply denoted the quantity $\mu_2(\infty)$ by μ_2. In this Section 1.6 I add "∞" as a time argument to avoid confusion.

[44] This is seen by verifying that

$$\mu_2(t-)\left(1-\delta\left(2;\mu_2(t-),m_2\right)\right)+\mu_1(t-)\left(\delta\left(1;\mu_2(t-),m_2\right)\right)=m_2.$$

Indeed, the left-hand side being the proportion of agents with a high valuation given an aggregate shock h_a, the idiosyncratic shocks defined by (1.39) and (1.40), and a suitable SLLN, whereas the right-hand side is the proportion of agents with a high valuation that was assumed in the first place.

and

$$\dot{P}_{c,t} \overset{(\Delta)}{=} \frac{\mathrm{d}P_{c,t}}{\mathrm{d}t}$$

being the time derivative of the function in (1.41). I also denote by

$$V(w, i\theta; m_2, t) \overset{\Delta}{=} \mathrm{E}\left[\int_t^{\infty} e^{-\rho s} U(\hat{c}_s)\,\mathrm{d}s \middle| \mu_2(t) = \mu_2(m_2, t)\right]$$

the value function of an investor having a liquid wealth of w, being of type i and holding θ units of the illiquid asset when a proportion $\mu_2(t)$ of the investors have a high valuation for the illiquid asset. I assume that the value function is differentiable in the time since the last shock, which is consistent with the assumption on the price process P_c.

My analysis of the dynamic setting is similar to the static analysis and is based on dynamic programming. I first derive the HJB equations for the dynamic problem. Then, I assume that the value function satisfies

$$V(w, i\theta; h_a, t) = -\exp\left\{-r\gamma\left(w + a(i\theta; h_a, t) + \bar{a}\right)\right\}, \tag{1.43}$$

with the constant

$$\bar{a} \overset{\Delta}{=} \frac{1}{r\gamma}\left(\frac{\rho}{r} - 1 + \log(r) + \gamma m_e\right).$$

This assumption is motivated by the static equilibrium analysis and justified *ex post.* Plugging the guess (1.43) into the HJB equation yields

$$\begin{aligned}
&r a(i\theta; m_2, t)\\
&= \sup_{\tilde{\pi}} \dot{a}(i\theta; m_2, t) + \kappa(i\theta; m_2, t; \tilde{\pi})\\
&\quad + \lambda_{i\bar{i}} \frac{e^{-r\gamma\left(a(\bar{i}\theta; m_2, t) - a(i\theta; m_2, t)\right)} - 1}{-r\gamma}\\
&\quad + 2\Lambda\mu\left(\bar{i}\bar{\theta}; m_2, t\right)\left[\frac{e^{-r\gamma\left(a(i\theta; m_2, t) - P_d(m_2, t)\left(\bar{\theta} - \theta\right) - a(i\theta; m_2, t)\right)} - 1}{-r\gamma}\right]^+\\
&\quad + \lambda_a \mathrm{E}^{m(\tilde{m}_2)}\left[\begin{array}{l} \delta(i; m_2, t; \tilde{m}_2)\frac{e^{-r\gamma\left(a\left(\bar{i}\theta; \tilde{m}_2, 0\right) + \tilde{\pi}\left(P_c\left(\tilde{m}_2, 0\right) - P_c(m_2, t)\right) - a(i\theta; m_2, t)\right)} - 1}{-r\gamma}\\ + \left(1 - \delta(i; m_2, t; \tilde{m}_2)\right)\frac{e^{-r\gamma\left(a\left(i\theta; \tilde{m}_2, 0\right) + \tilde{\pi}\left(P_c\left(\tilde{m}_2, 0\right) - P_c(m_2, t)\right) - a(i\theta; m_2, t)\right)} - 1}{-r\gamma} \end{array}\right]
\end{aligned}, \tag{1.44}$$

with

$$\begin{aligned}
\kappa(i\theta; m_2, t; \tilde{\pi}) \overset{\Delta}{=}& \theta m_d + \tilde{\pi}\left(\dot{P}_c(m_2, t) + m_c - r P_{c,}(m_2, t)\right)\\
&- \frac{1}{2} r\gamma \begin{pmatrix} 1 & \theta & \tilde{\pi} \end{pmatrix} \Sigma_i \begin{pmatrix} 1 & \theta & \tilde{\pi} \end{pmatrix}^*
\end{aligned} \tag{1.45}$$

representing the flow of mean-variance benefits resulting from holdings a certain portfolio, conditional on no illiquidity shock occurring.

The prices on the OTC market are still defined by the Nash bargaining solution. Adapting Proposition 4 from the static setting yields the price $P_d(m_2, t)$ bargained on the OTC market as the unique solution to the equation

$$\begin{aligned} &\eta_0\left(1 - e^{-r\gamma(a(20;m_2,t)-(a(2\Theta;m_2,t)-P_d(m_2,t)))}\right) \\ &\quad = \eta_\Theta\left(1 - e^{-r\gamma(a(1\Theta;m_2,t)-(a(10;m_2,t)+P_d(m_2,t)))}\right). \end{aligned} \tag{1.46}$$

The optimal policy and the resulting value function of the investors are characterized by the HJB equation (1.44). The impact of the illiquidity risk can be intuitively understood from this equation. The last line on the right-hand side represents the aggregate shocks. The random variable $\tilde{m}_2$ represents the proportion of agents with a high valuation after the shock, conditional on the occurrence of a liquidity shock.

In particular, this last line represents the utility shock expected by an investor when an aggregate shock occurs. The investor evaluates both the possibility that he may be directly affected by the shock, because her valuation may change, and the possibility of being affected by a change in the state of the economy. This change to the economy comes both from a price jump on the liquid market and from a change in the counter-parties on the OTC market.

When an investor chooses her holdings, she will take into account the mean-variance properties of the liquid asset and the covariance of this asset with her endowment. With aggregate shocks, however, she will also consider how the liquid asset hedges her own preference shocks and the shocks to her trading opportunity on the OTC market. The new dimension of the individual portfolio problem is the channel by which the illiquidity of the search market spills over and affects prices on the liquid market.

The HJB equation (1.44) characterizes the value function, but this characterization involves both partial derivatives and integrals of the value function. The general treatment of such an equation seems challenging. Instead, I focus on a certain asymptotic case.

More specifically, I let the agents become nearly risk-neutral with respect to the jump risks. In mathematical terms, this is done by letting the risk aversion go to zero,

$$\gamma \to 0, \tag{1.47}$$

and by scaling up the diffusion coefficients,

$$\begin{array}{ccccc} a_i & = & a_i(\gamma) & = & \frac{a_{i0}}{\sqrt{\gamma}} \\ b_i & = & b_i(\gamma) & = & \frac{b_{i0}}{\sqrt{\gamma}} \end{array}, \tag{1.48}$$

for constant numbers $\{a_{i0}, b_{i0}\}_i$ and an index $i \in \{1,2,c,d\}$. By doing so, the subjective quantity of risk

$$\gamma\Sigma_i(\gamma) = \gamma_0\Sigma_{i0}\,,\ i = 1,2$$

contained in the endowments and payout remains constant, even when investors become risk-neutral with respect to the risks driven by Poisson processes. These Poisson processes drive the random matching on the OTC market and the preference shocks.

The same approach is used to obtain closed-form expressions in Biais [1993], Duffie et al. [2007], or Vayanos and Weill [2008]. The procedure is particularly transparent in Gârleanu [2009]. This approach is also related to Skiadas [2013] and Hugonnier, Pelgrin, and Saint-Amour [2013]. In these models, there are several sources of risk and agents have a different level of risk-aversion for each risk. In my case, investors are risk-averse with respect to certain risks (diffusion risks) and risk-neutral with respect to other risks (jump risks).

Focusing on this asymptotic setting makes the analysis of aggregate demand shocks tractable.

Proposition 14. *There exists exactly one asymptotic equilibrium. In this equilibrium, the 2-investors buy the illiquid asset at all times if*

$$\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right)\cdot\det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right) > 0. \tag{1.49}$$

Furthermore, the difference of valuations for the illiquid asset

$$(a(2\Theta; m_2, t) - a(20; m_2, t)) - (a(1\Theta; m_2, t) - a(10; m_2, t))$$

is increasing in the quantity in (1.49) *and decreasing in the contact rate* Λ.

Proof. See Proof 50 in the Appendix. □

The proof of Proposition 14 relies both on algebraic manipulations of the equilibrium equations and, in a second step, on an application of Blackwell's sufficient condition for a contraction.

Proposition 14 indicates that the qualitative behavior of the dynamic equilibrium is similar to the behavior in the static setting. Specifically, the inequality (1.49) is the same as the inequality (1.33) defining the trading pattern in the static setting. Furthermore, the bargained price

$$P_d(m_2, t) = v(2; m_2, t) - \eta_{2l}\,(v(2; m_2, t) - v(1; m_2, t))$$

is the reservation value of a potential buyer subtracted by a share of the trade surplus. The share of the surplus is given by the bargaining power of the buyer. The trade surplus is, as stated in the last proposition, increasing in the risk-sharing made possible by the illiquid asset and decreasing in the contact rate on the OTC market. The contact rate reduces the trade surplus because it makes the search for a counter-party faster, improves the outside option of the investors, and reduces the benefits that one particular trade can bring. More generally, both Proposition 14 and its proof indicate that the intuition developed with the static model is robust to the introduction of aggregate liquidity shocks.

The aggregate demand shocks, however, create new effects in the model. More specifically, in the dynamic setting, the illiquidity of the OTC market affects prices on the liquid market. This spillover effect and, more generally, the returns on the liquid market are the object of the next proposition.

Proposition 15. *I assume that the inequality* (1.49) *holds, meaning that 2-investors buy the illiquid asset. Then, equilibrium expected excess returns on the liquid asset are*

$$\begin{aligned}
&\frac{1}{\mathrm{d}t}\left(\frac{\mathrm{E}\left[P_c(m_2, t+\mathrm{d}t)\,|\,(m_2,t)\right]}{P_c(m_2,t)} - r\right)\\
&\quad = \frac{m_d}{r} + o(\gamma)\\
&\qquad + r\gamma\begin{pmatrix} \frac{1}{P_c(m_2,t)}\left(S_c\Sigma_{cc} + \lambda_a \mathrm{E}\left[\left(P_{c,0}-P_{c,t}\right)^2\middle|(m_2,t)\right]\right)\\ +\frac{1}{P_c(m_2,t)}\left(S_d\Sigma_{cd}+\Sigma_{\eta c}\right)\\ +\lambda_a \mathrm{E}^{M(\tilde{m}_2)}\left[\left(\frac{P_c(\tilde{m}_2,0)}{P_c(m_2,t)}-1\right)\left(W(\tilde{m}_2,0)-W(m_2,t)\right)\middle|(m_2,t)\right]\end{pmatrix},
\end{aligned} \tag{1.50}$$

with

$$W(m_2,t) \triangleq \mathrm{E}^{\mu(i\theta;m_2,t)}\left[a(i\theta;m_2,t)\right]$$

being the average certainty equivalent across the population of investors. If the illiquidity $1/\Lambda$ *is small enough, these expected returns increase in the illiquidity* $1/\Lambda$ *when*

$$e_c\cdot(e_1-e_2) > 0 \tag{1.51}$$

and decrease otherwise.

Proof. See Proof 51 □

Proposition 15 offers a clean decomposition of the excess returns on the liquid asset into three different risk premia. The first two premia are classical. The first compensates investors for taking exposure to uncertain price movement and the dividend risk of the liquid asset. The second corrects the first by taking into account the diversification benefits against endowment risk and the dividend risk of the illiquid asset. The third premium is new and is driven by the illiquidity risk. It compensates investors for holding an asset that performs poorly exactly when trading on the OTC market becomes more difficult. To understand the underlying mechanism, let us first consider the average certainty equivalent $W(m_2, t)$. Intuitively, we can use the average certainty equivalent $W(m_2, t)$ to measure the efficiency of the allocation on the OTC market. Indeed, whenever the illiquid asset is transferred from a low valuation agent to a high valuation agent, there is a net gain in utility across the population, and $W(m_2, t)$ precisely reflects this utility gain.[45] As a result, $W(m_2, t)$ is a measure of the efficiency of the OTC market or, equivalently, of the reallocation speed on the OTC market.

Whenever there is a negative aggregate shock, meaning that the proportion of high-valuation investors drops, the imbalance on the OTC market is reduced, the search friction becomes more acute, and the OTC market becomes slower when it comes to reallocating the illiquid asset. When the inequality (1.51) holds, the negative aggregate shock induces a drop in the price of the liquid asset. As this price drop occurs precisely when trading OTC becomes more difficult, it commands a positive risk premium. In addition, this risk-premium increases with the intensity of the search friction.

Interestingly, this illiquidity spillover effect increases in the *level* of illiquidity $1/\Lambda$ but is driven by illiquidity *risk*. This can be readily seen from (1.50), where the impact of illiquidity on the expected returns stems from the "covariance"

$$\mathrm{E}^{m(\tilde{m}_2)}\left[\left(\frac{P_c(\tilde{m}_2, 0)}{P_c(m_2, t)} - 1\right)(W(\tilde{m}_2, 0) - W(m_2, t))\middle| (m_2, t)\right]$$

between the returns of the liquid asset and the efficiency of the economy. Furthermore, this covariance is scaled by the risk-aversion γ. The role of illiquidity risk can also be directly seen by comparing Proposition 15 with the static equilibrium described in Proposition 10. Indeed, in the static version of the model, there is no illiquidity risk and the frictions of the OTC market have no impact on the price of the liquid asset.

Conceptually, the spillover effect of Proposition 15 is similar to results in Acharya and Pedersen [2005]. Proposition 15, however, is based on an explicit modeling of illiquidity as a search friction and makes predictions regarding the sign of the illiquidity spillover effect. The model in Acharya and Pedersen [2005] relies on exogenous and stochastic transaction costs, and is thus more suited for illiquid but centralized markets. Furthermore, the price effect of illiquidity risk is driven by the exogenously specified covariance matrix of the transaction costs. Proposition 15 may thus be seen as a micro-foundation for the results in Acharya and Pedersen [2005]. My results also show that, unexpectedly, the measure of the illiquidity risk is determined by investors' certainty equivalent. It is interesting to compare Proposition 15 and the literature

[45] The proof of Proposition 15 contains a formal argument behind this statement.

on long run risks, pioneered by Bansal and Yaron [2004]. In this literature, the assumption of recursive (non time separable) preferences implies that the risk premia are determined by investors' certainty equivalent. This channel implies that, in stark contrast to the case of time separable preferences, long run risk is priced in today's returns. In my model, Proposition 15 shows that, in illiquid markets, long run risk is priced despite the fact that agents have standard, time separable preferences. This interaction between long run risk and illiquidity is an interesting and important topic for future research.

Proposition 15 can also be used to evaluate and improve empirical analysis of illiquidity. Indeed, following the example of Longstaff et al. [2005], a number of authors willing to measure the illiquidity component of bond yields have considered the so-called CDS basis, defined as the spread between bond excess returns and CDS premia. The rationale for this procedure is the relatively high liquidity of CDS markets when compared to bond markets. In particular, CDS spreads should be a clean measure of credit risk.[46] As Proposition 15 indicates, however, even the returns on a perfectly liquid market may be affected by the illiquidity of a related market.

Finally, Propostion 15 is consistent with empirical findings regarding illiquidity spillover. For example, Tang and Yan [2006] and Lesplingart, Majois, and Petitjean [2012] document how the illiquidity of the bond market increases yields on CDS contracts, which is exactly in line with Proposition 15. Das and Hanouna [2009] documents a similar effect between stock and CDS markets.

1.7 Opening the Liquid Market

In this section, I consider the effect of the liquid market on the functioning of the illiquid one. For tractability reasons, I again focus on the stationary setting of Section 1.5.

As Proposition 10 and Proposition 14 show, the trading pattern on the OTC market is determined by the quantity

$$\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right).$$

This quantity measures how much risk-sharing can be achieved on the OTC market only and, as a result, is closely linked to the competitive price of the asset traded OTC. At the same time, the Nash bargaining solution assumed for the OTC market also makes this quantity a measure of the illiquidity discount.[47] In particular, if the liquid asset mitigates the search friction and

[46] Illiquidity is also priced on CDS market, as shown by Bongaerts et al. [2011] and Junge and Trolle [2013]. It is, however, true that CDS markets are typically significantly more liquid then their underlying bond markets.

[47] See Appendix A for explicit derivations supporting these claims.

decreases the illiquidity discount on the asset traded OTC, the liquid asset will necessarily also decrease the competitive price of the asset. This *mitigation* effect occurs because the liquid asset offers an attractive alternative to the asset traded OTC. However, this alternative market also diverts some of the fundamental demand for the asset traded OTC, leading to a *capture effect*. The mitigation effect tends to increase prices on the OTC market whereas the capture effect tends to decrease them. In this section I evaluate which of these effects dominates. I do so by comparing the economy with and without the liquid market.

In the real world, the decision to create a market for a new security is always endogenous and is determined by the financial intermediaries' estimates of the investors' trading needs. These intermediaries can be dealers, who will make the market in the new security, or the exchanges on which the security will be traded.[48] The revenues of these intermediaries are driven by trading volumes, and so is financial innovation. As a result, I assume that the liquid asset is designed to maximize volumes.[49]

I derive the trading volumes for the proof of Proposition 12 (see Appendix D). Shares of the liquid asset are exchanged at the rate

$$V = \frac{1}{\Sigma_{cc}} \left(|\Sigma_{cd}| \Theta 2 \Lambda \mu (1\Theta) \mu (20) + |\Sigma_{1c} - \Sigma_{2c}| \frac{\lambda_{12}\lambda_{21}}{\lambda_{12} + \lambda_{21}} \right) \tag{1.52}$$

when the 2-investors buy the illiquid asset.[50] Recalling the definition of the covariation matrices in (1.14) and defining the constants

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \triangleq \Theta 2 \Lambda \mu (1\Theta) \mu (20) \begin{pmatrix} a_d \\ b_d \end{pmatrix}$$
$$\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \triangleq \frac{\lambda_{12}\lambda_{21}}{\lambda_{12} + \lambda_{21}} \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix},$$

48 Regarding a description of financial innovation being driven by intermediaries rather than by end-users, one may refer to Das et al. [2013]:

> CDS introduction is initiated by dealer banks depending on factors such as size of outstanding debt of an issuer, underlying credit risk of the issuer, and demand for credit protection. [...] Introduction of an equity option is decided by the corresponding options exchange depending upon factors such as trading volume, market capitalization and turnover of the underlying stock.

Duffie and Jackson [1989] proposes a model of financial innovations by intermediaries who maximizes trading volumes. Rahi and Zigrand [2009] and Rahi and Zigrand [2010], for instance, propose alternative theoretical models of financial innovation driven by intermediaries.

49 I do not model the intermediary explicitly. However, an intermediary who earns a constant bid-ask spread on transactions will attempt to maximize trading volumes. And the trading volume with a constant bid-ask spread converges toward the volume without transaction costs when the bid-ask spread decreases. See Praz [2013] for a treatment of transaction costs in a setting similar to the one of this paper.

50 The expression (1.52) also describes the asymptotic trading volume in a setting with aggregate demand shocks when the risk-aversion γ goes to zero. See the characterization of the asymptotic optimal liquid holdings in the proof of Proposition 14.

I rewrite Equation (1.52) as

$$\Sigma_{cc} V = |w_1 a_c + w_2 b_c| + |w_3 a_c + w_4 b_c|.$$

When 1-investors buy the illiquid asset, the only change is that the weights w_1 and w_2 must be rescaled.[51]

It is important to understand which model parameters influence the equilibrium trading volume. First, the trading volume V is independent of the expected payout of the liquid asset, as seen in Equation (1.52). This expected payout must thus be set exogenously. Second, the number of shares exchanged can be made arbitrarily large by scaling down the risk exposures e_c of the liquid asset.[52] Without loss of generality, I normalize the overall exposure

$$\Sigma_{cc} \quad \left(= \|e_c\|_2^2 = a_c^2 + b_c^2\right)$$

to 1.

Summing up, I choose the liquid asset that maximizes the trading volume, meaning that I choose the risk-profile e_c of the liquid asset to be a point of maximum in the optimization

$$\max_{(a_c, b_c)} \{|w_1 a_c + w_2 b_c| + |w_3 a_c + w_4 b_c|\} \tag{1.53}$$

under the conditions

$$\|(a_c, b_c)\|_2 = 1,$$

$$\det\left(\left(\begin{array}{c|c} a_d & a_c \\ b_d & b_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c|c} a_1 - a_2 & \alpha_c \\ b_1 - b_2 & \beta_c \end{array}\right)\right) > 0. \tag{1.54}$$

Three features of the maximization (1.53) should be emphasized. First, the constraint (1.54) ensures the consistency of the beliefs regarding the trading pattern on the OTC market. Specifically, as the objective function (1.53) assumes that 2-investors buy the asset traded OTC, the inequality (1.54) ensures that this assumption is justified *ex post*.

Second, the inequality (1.54) is strict. The borderline with equality corresponds to the case in which all investors have the same reservation value for the illiquid asset. In this case, the

[51] The rescaling factor is

$$\frac{\mu(2\Theta)\mu(10)}{\mu(1\Theta)\mu(20)}$$

and is the ratio of the type flows generated by trading when 1-investors buy the illiquid asset and when 2-investors do.

[52] Dividing e_c by $K > 0$ multiplies the trading Volume by K.

benefits resulting from any trade on the OTC market are zero, investors are indifferent on the OTC market, and I would need more assumptions to define the type dynamics and the trading volumes.[53]

Third, both the objective function and the domain of optimization in (1.53) are symmetric around the origin. As a result, whenever a point x one the unit circle is a point of maximum, so is its opposite $-x$.

I can characterize the solution to the optimal asset design problem (1.53).

Proposition 16. *There exists a solution to the volume maximization* (1.53) *exactly when*

$$\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}\right) \times \left(\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}\right) < 0. \tag{1.55}$$

In this case, the optimal liquid asset is

$$\begin{pmatrix} a_c \\ a_d \end{pmatrix} = \pm \frac{1}{\nu} \begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}, \tag{1.56}$$

with the constant

$$\nu \triangleq \sqrt{(w_1 - w_3)^2 + (w_2 - w_4)^2}$$

ensuring the normalization $\Sigma_{cc} = 1$.

Proof. See Proof 48 in the Appendix. □

Proposition 16 implies that the optimal liquid asset is the weighted average of two risk profiles. The first is the profile of the illiquid asset and the second is optimal in terms of risk-sharing. This already indicates how the new liquid asset balances the attempt to capture some of the trading activity that takes place OTC and the alternative aim of being valuable to as many investors as possible. Furthermore, the weight on the profile of the illiquid asset is monotone increasing in the contact rate on the OTC market because, with a higher contact rate, there is more volume to capture. Perhaps paradoxically, this means that the search friction is easier to mitigate when it is smaller in the first place.

Importantly, the optimal security design defined by the maximization (1.56) does not necessarily have a solution. In particular, if inequality (1.55) does not hold and the liquid asset has the risk-profile in Equation (1.56), then 1-investors have the higher valuation for the illiquid asset. Conversely, if the trading volumes had been optimized under the assumption that 1-investor

[53]The additional assumptions would require to randomize the decision to trade.

buy the illiquid asset, the resulting liquid asset would actually induced the 2-investors to buy the illiquid asset. As a result, the only way of, possibly, obtaining an optimum would be to impose the behavior of the investors on the OTC market when investors are indifferent.

Finally if the trading pattern on the OTC market is the same before and after the opening of the liquid market, the type flows across the population will not change when the liquid asset is introduced. As a result, the trading volumes will be constant at any time. However, if the opening of the liquid market inverts the trading pattern on the OTC market, the trading volume (1.52) only represents the asymptotic trading volume in the steady state.[54]

Intuitively, choosing a liquid asset that is very similar to the illiquid asset has two consequences. On the one hand, the liquid asset mitigates the search frictions and reduces the illiquidity discount on the OTC market. This pushes the price on the OTC market up. On the other hand, if the liquid asset is very similar to the illiquid asset, the illiquid asset has little value left as a risk-sharing instrument. This second effect pushes the price of the illiquid asset down.

Figure 1.4 illustrates how each of these effects can dominate. When the search friction is severe, on the right part of the plot, the mitigation of the illiquidity discount dominates the drop in the fundamental value and the price on the OTC market increases when the liquid asset starts trading. Quite differently, when the search frictions are modest, on the left part of the plot, the illiquidity discount is small and diversion of trading volume towards the new market dominates the benefits of the new hedging opportunities. To complete this section, I note that, for potential applications to bond market, it may be more natural to consider the yield on the illiquid asset. This is done in the second panel of Figure 1.4.

1.8 Conclusion

I study a general equilibrium model in which agents can trade both on an illiquid OTC market and on a liquid, centralized market. Search frictions on the OTC market increase the trading volume and open-interest on the liquid market. Furthermore, the endogenous interactions of the search frictions with the aggregate demand shocks generate a time-varying efficiency of the asset allocation on the OTC market. This liquidity risk is priced and affects the risk premium on the liquid asset. These results are consistent with a number of empirical studies.[55]

Motivated by several real-world examples in which centralized markets were created as an alternative to preexisting OTC markets, I introduce endogenous financial innovation into the

[54] See Theorem 5 (and its proof) in Duffie et al. [2005] for a similar issue. Namely, Duffie et al. [2005] show how, for a sufficiently patient intermediary, an optimal policy chosen at time zero and fixed afterwards is approximately the policy that maximizes revenues in the steady state.

[55] Regarding the interactions between CDS and bond markets see, for example, Oehmke and Zawadowski [2013], Tang and Yan [2006], Lesplingart et al. [2012], or Das and Hanouna [2009].

model. I assume that intermediaries design the cash flows of the liquid asset that maximize the equilibrium trading volume. Then, I compare the prices on the OTC market in an economy with and without a liquid asset. I show that the risk profile of the optimal liquid asset is a weighted average of the illiquid asset's profile and of the risk profile that would lead to an efficient risk-sharing. The weight on the profile of the illiquid asset is shown to be monotone increasing in the contact rate on the OTC market because, with a more active OTC market, there is more trading volume to capture.

I show that the liquid market has two effects on the illiquid OTC market. On the one hand, it mitigates the search frictions, reduces the price discount on the illiquid asset, and increases the prices bargained on the OTC market. On the other hand, the liquid asset captures some of the illiquid asset's value as a risk-sharing instrument. I show how each of these effects can dominate, and link this equilibrium behavior with the empirical literature studying how the onset of CDS trading affects bond yields.

I believe that understanding the role of liquidity in portfolio selection and its general equilibrium feedback effects is both important and timely. Several regulatory reforms such as the Dodd-Frank Act in the US and the MiFID II proposal in the European Union propose to significantly revise the functioning of modern markets and, in particular, to move some of the OTC trading to centralized exchanges. The only way to evaluate the consequences of these reforms is to develop a general equilibrium model that accounts for the trading frictions on OTC markets and their cross-market externalities.

My model could be enriched in several directions. For example, throughout this paper, I assumed a dichotomy between an illiquid OTC market and a perfectly liquid market. In many real-world examples, however, the alternative to a costly search process will be to trade on another market immediately, but at a cost. In Praz [2013] I introduce this additional liquidity friction, and consider a general equilibrium model in which investors balance transaction costs and execution uncertainty when they select their portfolio holdings. Financial intermediaries rationally anticipate this behavior and optimally choose the bid-ask spreads on the exchange, the level of liquidity provision on the OTC market, and the form of financial innovation. Finally, introducing asymmetric information, either in terms of common value uncertainty, as in Duffie, Malamud, and Manso [2009], or in terms of private liquidity needs would also significantly enrich the structure of the model. A model of OTC market with asymmetric information would shed light on the current regulatory debates aiming at increasing the transparency of OTC markets. We take some first steps in this direction in Cujean and Praz [2013].

1.9 Figures

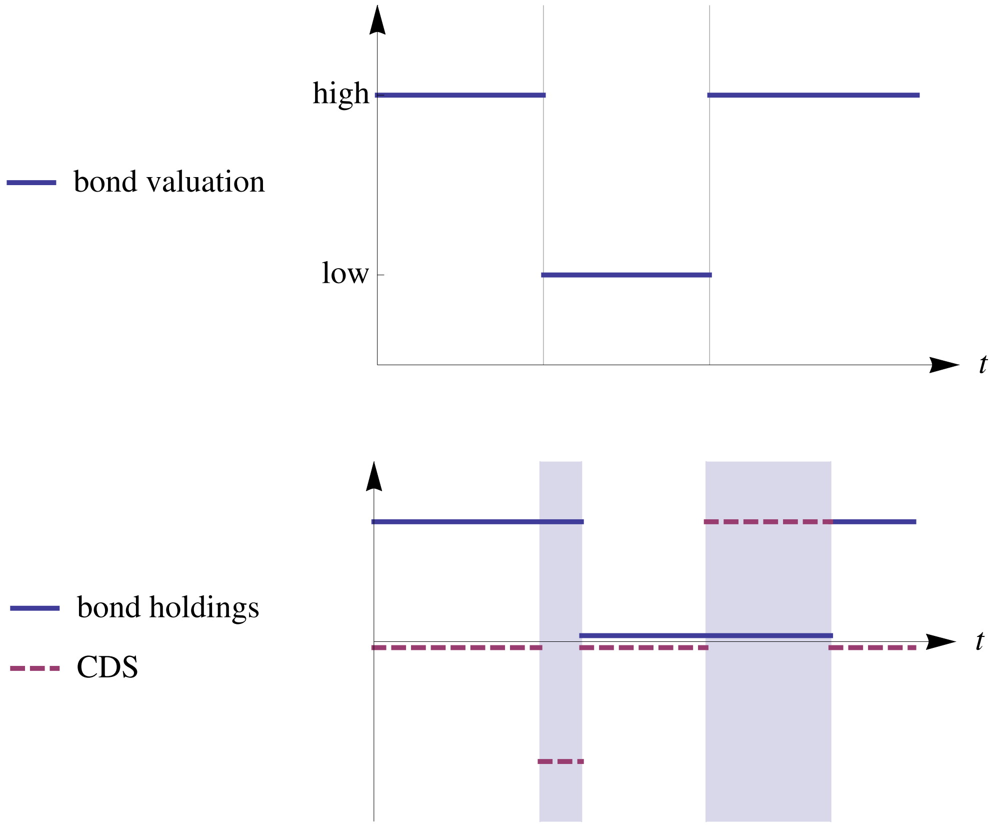

Figure 1.1: The first panel represents the subjective valuation of an asset traded OTC, say a bond, by a given investor and how this valuation changes over time. The second panel represents this investor's holdings in the illiquid asset (solid line) and a more liquid security offering a similar exposure (dashed line). If the illiquid asset is a bond, the liquid security could be a CDS (as a protection seller). The shaded areas corresponds to the periods during which the investor is searching for a counter-party on the OTC market. During these periods, the investor hedges her sub-optimal exposure to the illiquid asset by trading the liquid asset. These plots are illustrative and not based on the parameters in Table 1.1.

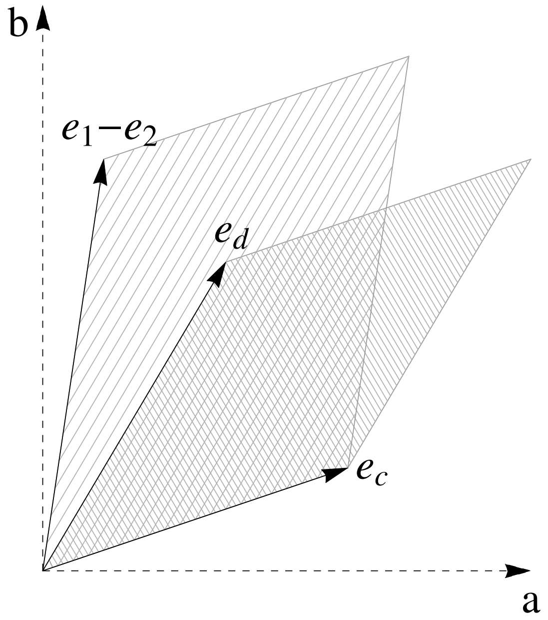

Figure 1.2: This plot represents the vector of exposures of the liquid asset (e_c) and of the illiquid asset (e_d), along with the differences of the exposures between the two types of investors ($e_1 - e_2$). The horizontal axis measures the exposure to the aggregate risk a and the vertical axis measures the exposure to the aggregate risk b. The surface of the quadrangle with the narrow dash is $|\det(e_d : e_c)|$ and measures how orthogonal the exposures of the two assets are. The surface of the quadrangle with the broad dash is $|\det(e_1 - e_2 : e_c)|$ and measures how orthogonal the risk profile of the liquid asset is to the profile that would be optimal in terms of risk-sharing. The two quadrangles intersect when $\det(e_d : e_c)\det(e_1 - e_2 : e_c) > 0$, which is the condition appearing in Proposition 10 and defining the trading pattern on the OTC market.

Table 1.1: Baseline parameter values.

notation	parameter	value
S_c	supply of the liquid asset	0
S_d	supply of the asset traded OTC	0.8
η_Θ, η_0	bargaining powers	$\frac{1}{2}$
λ_{21}	arrival rate of idiosyncratic liquidity shocks	$\frac{1}{5}$
λ_{12}	recovery rate from a liquidity shocks	5
Λ	meeting rate	50
r	risk-free rate	0.037
m_c	expected payouts of the liquid asset	0.05
m_d	expected payouts of the asset traded OTC	0.05
(a_c, b_c)	exposures of the liquid asset	$(1.0000, -0.0016)$
(a_d, b_d)	exposures of the asset traded OTC	$(0.1022, -0.0002)$
(a_1, b_1)	exposures of the endowment for the investors of type 1	$(9.4718, -0.0150)$
(a_2, b_2)	exposures of the endowment for the investors of type 2	$(-0.5017, 0.0008)$
γ	coefficient of absolute risk aversion	2
Θ	holdings in the asset traded OTC	1

Choice of the parameters The supply of the illiquid asset, the holdings size Θ, and the dynamics of the idiosyncratic shocks are taken from Duffie et al. [2007]. The liquid asset is understood to be a derivative and its net supply is zero. The risk-free rate and expected payouts of the assets are the same as in Gârleanu [2009] (the calibration in Gârleanu [2009] is itself based on Campbell and Kyle [1993] and Lo et al. [2004]). The baseline meeting intensity is within the standard range and corresponds an average of one meeting per week. The risk-aversion is chosen within the standard range. The exposures of the assets and endowments are chosen to satisfy the following conditions.

1. The profile of the illiquid asset maximizes the reservation value of the illiquid asset in a setting without liquid asset, conditionally on the exposures of the endowments. This maximization captures, in a reduced form, the strong clientèle effects on OTC markets.

2. The two risky assets should have an expected return of approximately 5%.

3. The 2-investors buy the illiquid asset both before and after the opening of the liquid market. See the discussion after Proposition 16.

4. The two risky assets should have the same price for the baseline parameter values. This is for ease of visualization only.

The values of the asset and endowment exposures are then found numerically.

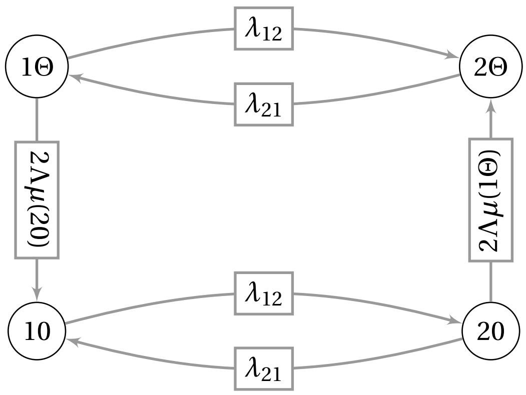

Figure 1.3: Each vertex is a type of investors (for a type $i\theta$, $i \in \{1,2\}$ is the type of exposure and $\theta \in \{0,\Theta\}$ are the holdings in the illiquid asset). Each arrow indicates a flow between types and the number on each arrow is the corresponding transition intensity for a given investor. These flow are valid under Assumption 6 and corresponds to the flow equations (1.25).

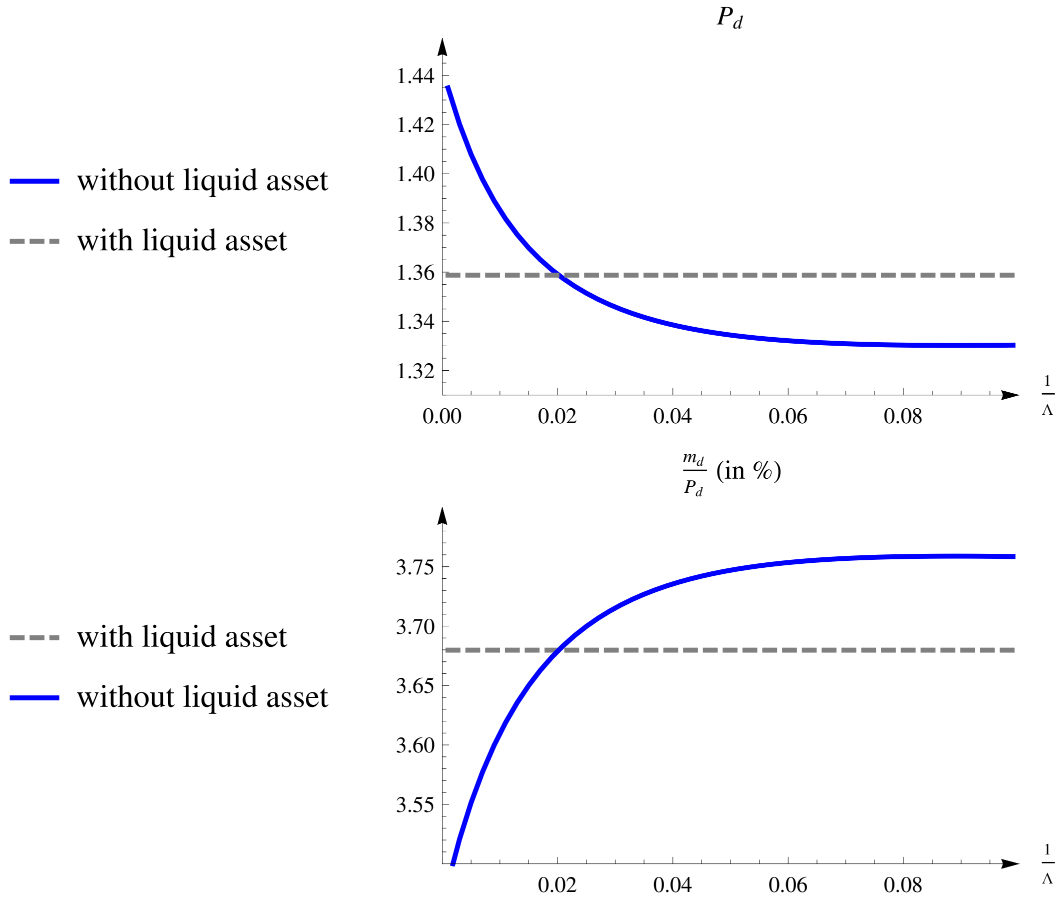

Figure 1.4: The upper panel is a plot of the price bargained on the OTC market as a function of the meeting intensity on the OTC market. The continuous line is the price when the OTC market is the only market in the economy, whereas the dashed line is the price when investors can trade both on the liquid and on the OTC market. The lower panel plots the expected returns on the illiquid assets, but is otherwise similar to the upper one. The parameter values for this plot are in Table 1.1.

2 Pay or Wait: Equilibrium Prices with Two Trading Frictions

I[1] study an equilibrium model in which investors hedge endowment risk on two markets. The first market is centralized and trades can be executed at any time but against proportional transaction costs. The second market is over-the-counter (OTC) and requires to search for a counter-party. I show the existence of an equilibrium for any level of transaction costs and characterize in closed-form how the two frictions jointly modify the risk-premia on both markets. The transaction costs reduce both the transaction sizes and frequencies on the centralized market, and affect the price on the OTC market by making hedging more costly. The prices bargained OTC depend non-monotonically on the transaction costs on the exchange. I link my results to recent regulatory proposals and show how taxing CDS trading may both increase of decrease the cost of debt financing.

2.1 Introduction

Ongoing political discussions in the EU aim at taxing financial transactions.[2] These taxes would apply to most financial transactions, including credit default swap (CDS) trades, but debt securities may be excluded.[3] Confirming this pattern, the French transaction tax introduced in 2012 affects CDS transactions but not debt transactions. The discussions regarding financial transaction taxes thus appear to rely on the assumption that taxing CDS trading but

[1] I thank my advisor Semyon Malamud for his continuous guidance. I also thank Pierre Collin-Dufresne, Julien Cujean, Nicolae Gârleanu, Julien Hugonnier, and Ramona Westermann for their comments and suggestions. I also thank participants to the 6th Erasmus Liquidity Conference in Rotterdam.

[2] These proposals are clearly designed as punitive actions. A 2011 press release of the European Commission was entitled “Financial Transaction Tax: Making the financial sector pay its fair share”. In the US, a bill proposed in 2009 is called “Let Wall Street Pay for the Restoration of Main Street”

[3] For instance, Bloomberg Businessweek (Rebecca Christie and Jeanna Smialek) [April 21, 2013] reports a statement by Ferdinando Nelli Feroci, the Italian Permanent Representative to the European Union in Brussels at the time. Talking about financial transaction tax, Mr. Nelli Feroci claimed that “Transactions on government bonds must be excluded” and that this point is “not up to negotiation”.

not bond trading will decrease the cost of debt financing.[4]

In an empirical study, Oehmke and Zawadowski [2013] show how investors use CDS and bond markets as two alternative means of adjusting their exposure to the same fundamentals. Das et al. [2013] also provides some evidence in this sense. What distinguishes the two markets is liquidity. Indeed, even if both the CDS and bond markets are over-the-counter (OTC), the search frictions on the CDS market are far less severe than the frictions on the bond market.[5] The difference in search frictions reflects the difference in market fragmentation, with the CDS market being a uniform, "one size fits all" alternative to the the many bonds, with many maturities, embedded options, and covenants, available for each issuer.

Evaluating the effect of taxing CDS transactions on bond yields thus requires to understand the portfolio decisions of investors facing two types of illiquidity. These two types are, on the one hand, an explicit cost paid to execute a transaction immediately and, on the other hand, the costly uncertainty of having to search for a counter-party on an illiquid OTC market.

Analyzing the interaction between these two types of illiquidity is relevant beyond the discussion of financial transaction taxes. Indeed, even in the absence of taxes, investors pay a bid-ask spread to the CDS dealer for providing immediacy on the CDS market and, by doing so, make it possible to bypass the frictions of the bond market. Further examples of a trade-off between immediacy and uncertainty abound. One may think of an investor trading collateralized debt obligations instead of loans, or a property total return swap instead of real estate assets. Further example comparing immediacy and uncertainty include the "upstairs" and "floor" stock markets, or the comparison of limit and market orders in limit order books. Finally, investors face a similar trade-off when they balance the costly convenience of bespoke derivative and structured products against the basis and operational risks of a dynamic replication strategy.

The problem of an investors balancing the costs of immediacy against execution uncertainty is complex. The investor must consider the explicit transaction costs, the implicit search costs, and the basis risk between the two assets. The exact effect of the frictions on the individual policies and how these effects aggregate can only be analyzed in a general equilibrium setting. I propose such a setting in this paper.

I study an economy in which risk-averse investors share endowment risk by trading two imperfectly correlated assets. The first one is traded on an exchange, with a constant bid-ask spread, whereas the second is traded on on illiquid OTC market. I follow Duffie et al. [2007] for the modeling of the OTC market. In this framework, investors meet randomly at a given rate

[4] See, for instance, The Financial Times (Tracy Alloway) [November 6, 2011] and The Financial Times (opinion by Andrew Baker) [November 9, 2011] for reports of the popular criticisms against CDS trading and its impact on bond markets

[5] See Longstaff et al. [2005] for a discussion of the relative liquidity of the CDS and bond market.

and the Nash bargaining solution characterizes the bilateral trades. Liquidity, measured by the meeting rate, affects investors both because contacting a potential trading partner is time consuming and because prices are not competitive. The interaction between the two markets is driven by the investors who use the exchange as a substitute for the OTC market and hedge on the exchange while searching for a counter-party on the OTC market.

I show the existence of an equilibrium for any level of transaction costs and obtain closed-form expressions for the equilibrium quantities. The existence argument is non standard because the transaction costs introduce discontinuities in the equilibrium equations.

The search frictions on the OTC market affect the price of the asset on the exchange. As made clear by Gârleanu [2009], the equilibrium effect of transaction costs depends on the imbalance between the investors who last purchased and those who last sold the asset. When the meeting rate on the OTC market varies, so does the rate at which investors hedge (by both buying and selling) on the exchange, and whether the buying or selling pressures dominates depends on the dynamics of the hedging demand. I show that, when low valuation spells are relatively short, the search friction on the OTC market increase the risk premium on the exchange.

The transaction costs on the exchange have two main effects. First, they make the marginal cost of trading on the exchange higher. Investors optimally react to these costs by reducing the size of their transactions *when they trade*. Second, the transaction costs also make trading on the exchange less frequent. Indeed, when an investors receives a shock to her hedging demand, she balances the costs of trading on the exchange, the optimal risk-profile that she may achieve by trading, and when this risk-profile will have to be adjusted again. When the transaction costs are high enough, the round-trip transaction costs can dwarf the improvement to the risk-exposure and investors optimally decide not to trade after certain idiosyncratic shocks.

As the transaction costs increase the costs of hedging, they also affect the outside options of the agents bargaining on the OTC market. As I show with numerical examples, the relationship between the transaction costs on the exchange and the price bargained on the OTC market is non-monotonic. The intuition is as follows.

As long as the transaction costs are low enough, investors on both the long and the short side of the market hedge on the exchange while searching for a counter-party on the OTC market. As the investors on the short-side of the market find a counter-party more easily, these investors trade at a higher frequency. As a result, any increase of the transaction costs is particularly detrimental to the investors on the short-side of the market and makes their outside option particularly weak.

It is standard to assume that sellers are on the short side of the market.[6,7] In this case, increasing the transaction costs on the exchange makes the sellers relatively weaker when they bargain, and the asset traded OTC is exchanged for a lower price.

If the transaction costs increase beyond a certain threshold, however, investors on the short side of the market stop hedging while they are searching for a counter-party. As a result, any further increase of the transaction costs mostly affects the investors on the long side of the market and make their outside options weaker when thy bargain. As a result, when buyers are on the long side of the OTC market and when the transaction costs are sufficiently large, the price bargained OTC is increasing in the transaction costs on the exchange. As I show in numerical examples, the price bargained with sufficiently high transaction costs can be higher than when these costs are zero.

I finally come back to the impact of taxing CDS trading on bond yields. The equilibrium behavior of my model indicates that making CDS trading costly may indeed decrease bond yields. The equilibrium relationship between taxes and yields is, however, non-monotonic, and depends in a rather subtle way on the strength of the trade motives, their dynamics, and the relative levels of the two types of illiquidity. A financial transaction tax may thus decrease debt financing, but the exact effect is hard to predict. Introducing a financial transaction tax to decrease borrowing yields seems to be a hazardous strategy.

Literature Review My paper is related to four main strands of literature. First, I use the framework introduced in Duffie et al. [2005] and Duffie et al. [2007] to model the OTC market. A number of references rely on related settings. Examples include Weill [2007], Weill [2008], Gârleanu [2009], Afonso and Lagos [2011], Lagos and Rocheteau [2007], Lagos and Rocheteau [2009]. These references all analyze a unique illiquid market, whereas I look at the interactions between two illiquid markets.

Second, my paper is related to the literature comparing market structures or market liquidity levels. For instance, Biais [1993], De Frutos and Manzano [2002], Yin [2005] compare centralized and fragmented markets in a static setting. In Pagano [1989], Rust and Hall [2003], and Miao [2006], investors can choose between a centralized and a search market, but investors only trade once and, as a result, there is no cross-market hedging.

[6] This is, for instance, Condition 1 in Duffie et al. [2005], Inequality (6) in Weill [2007], and Assumption 2 in Vayanos and Weill [2008]. When this assumption does not hold, search frictions increase the price of illiquid assets, and effect that is only rarely observed on actual markets. An exception may be the on-the-run Treasuries, see the discussion in Duffie et al. [2007].

[7] If buyers are on the short side of the market, the search frictions increase prices, a phenomenon that is rarely observed on actual markets. An exception may be the on-the-run Treasuries. See the discussion in Duffie et al. [2007].

My paper is also related to the literature in asset pricing that evaluates the equilibrium effects of trading frictions. Examples focusing on transaction costs include Constantinides [1986], Lo et al. [2004], Vayanos [1998], Huang [2003], and Garleanu and Pedersen [2013]. More recent recent examples that draw links with current regulatory reforms include Buss and Dumas [2013] and Buss, Dumas, Uppal, and Vilkov [2013]. Some other references consider markets with different levels of search frictions. These references include Vayanos and Wang [2007] and Vayanos and Weill [2008]. I study the interaction between transaction costs *and* search frictions in a dynamic setting.

Finally, I analyze the interaction between costly trading immediacy and uncertainty regarding the trade execution. This issue is related to the choice of market against limit order in limit order book markets. Foucault, Kadan, and Kandel [2005], Parlour [1998], Goettler, Parlour, and Rajan [2005], and Roşu [2009], for instance, analyze the dynamics of limit order books.

2.2 Model

The model in this section is essentially the same as in the first chapter of my dissertation, except that trading on the exchange is costly and requires to pay proportional transaction costs. For convenience, I recall the main elements of the model.

I study an economy in which investors share endowment risk by trading two different assets on, respectively, a liquid exchange and an OTC market. This model is an extension of Duffie, Gârleanu, and Pedersen [2007].

Assets and investors Two independent aggregate risk factors are described by the Brownian motions

$$\left(B_{a,t}, B_{b,t}\right)_{t\geq 0}.$$

Two risky assets, c and d, are exposed to these risk factors. The cumulative dividend payouts of these assets satisfy

$$\begin{aligned} \mathrm{d}D_{c,t} &= m_c\,\mathrm{d}t + a_c\,\mathrm{d}B_{a,t} + b_c\,\mathrm{d}B_{b,t}, \\ \mathrm{d}D_{d,t} &= m_d\,\mathrm{d}t + a_d\,\mathrm{d}B_{a,t} + b_d\,\mathrm{d}B_{b,t}. \end{aligned} \tag{2.1}$$

These assets are available in net supplies S_c and S_d, respectively. As described below, the asset c is traded on a *c*entralized market, whereas the asset d is traded on a *d*ecentralized, OTC market. For convenience, I define the vectors

$$e_c \triangleq \begin{pmatrix} a_c \\ b_c \end{pmatrix},\ e_d \triangleq \begin{pmatrix} a_d \\ b_d \end{pmatrix}$$

and call them the exposures of the assets c and d, respectively.There is also a risk-free asset, available in perfectly elastic supply, and paying out dividends at the constant rate $r > 0$.[8]

The economy is populated by a continuum of investors. I write μ for a normalized measure over this continuum. Each investor receives an endowment driven both by the aggregate risk factors and by idiosyncratic shocks. More specifically, the cumulative endowment of investor i satisfies

$$d\eta_t = m_\eta\, dt + a_{i,t}\, dB_{a,t} + b_{i,t}\, dB_{b,t}, \tag{2.2}$$

and is thus driven by the two aggregate risk factors. The vector of exposures

$$e_{i,t} \triangleq \begin{pmatrix} a_{i,t} \\ b_{i,t} \end{pmatrix} \tag{2.3}$$

evolves stochastically. Specifically, the stochastic vector $e_{i,t}$ is a time-homogeneous Markov chain jumping back and forth between two (two-dimensional) values.[9] These two values are

$$e_1 \triangleq \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \text{ and } e_2 \triangleq \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad (\in \mathbb{R}^2)$$

and I denote by

$$\begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{pmatrix} \tag{2.4}$$

the generator of the Markov chain. The Markov chains are independent across agents.

Trading mechanisms Investors trade the liquid asset c on a centralized market and any trade requires the payment of proportional transaction costs. The exact impact of these costs is discussed below. Investors also trade the risk-free asset at any time and without costs.

The other risky asset, d, is traded OTC. Trading d thus requires searching for a counter-party and negotiating the details of the transaction. The search process is governed by a "random

[8] The interest rate is exogenous in all the models of asset pricing with search that I am aware of.

[9] In particular, both components of the vector of exposures jump together.

matching". That is, a given investor gets in touch with another investor at the jump times of an idiosyncratic Poisson process with intensity Λ. This other investor is randomly drawn from across the population of investors. The draws are independent across meetings.

As the meeting intensity Λ controls the search friction on the OTC market, I call it the *liquidity* of the OTC market. Given the dynamics of a Poisson process, the inverse

$$\xi \triangleq \frac{1}{\Lambda}$$

of the liquidity is the expected search time until the next meeting. I call this expected time the *illiquidity* of the OTC market.

Taking things together, investors from two separate subsets B and C of the population meet at the rate

$$2\Lambda\mu(B)\mu(C), \tag{2.5}$$

with μ being a measure on the set of investors.

Once two agents have met, they bargain over a possible trade in the illiquid asset d, and the outcome of the bargaining is given by the Nash bargaining solution.

Preferences Each investor i maximizes her expected utility from consumption. Her utility function U has a constant coefficient of absolute risk aversion $\gamma > 0$ (exponential or CARA utility), meaning that

$$U : c \mapsto -e^{-\gamma x},$$

and their subjective rate of discounting is $\rho > 0$. The consumption and investment policy of i is thus dictated by the optimization

$$\sup_{\tilde{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(\tilde{c}_u)\, \mathrm{d}u \middle| \mathscr{F}_{i,0}\right], \tag{2.6}$$

with the admissible consumption processes $\tilde{c}$ satisfying certain conditions defined below and $\mathscr{F}_{i,0}$ being all the information available and relevant to i at time 0.

The payouts of the risky assets, defined in (2.1), are independent and identically distributed across time. Furthermore, the idiosyncratic exposure shocks defined by (2.4) offer a unique and stable stationary distribution of types 1 and 2 across the population. As a result, I expect all the aggregate quantities to be constant in the long run and I focus my analysis on this asymptotic, stationary case.[10] In a stationary equilibrium, the information set $\mathscr{F}_{i,0}$ only contains idiosyncratic quantities and the individual problem (2.6) becomes

$$V(w,t) \triangleq \sup_{\tilde{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(\tilde{c}_u)\,\mathrm{d}u \,\middle|\, \begin{array}{ccc} w_0 & = & w \\ t_0 & = & t \end{array}\right], \tag{2.7}$$

with w_0 being the wealth invested by i at time zero in the risk-free asset and t_0 being i's type at time zero. The exact definition of the type in the setting with transaction costs is discussed below.

Budget constraint The consumption and trading of an investor must be consistent with the dynamics of her wealth. Specifically,

$$\mathrm{d}\tilde{w}_t = r\tilde{w}_t\,\mathrm{d}t - \tilde{c}_t\,\mathrm{d}t + \mathrm{d}\eta_t + \tilde{\theta}_t\,\mathrm{d}D_{dt} - P_d\,\mathrm{d}\tilde{\theta}_t + \tilde{\pi}_t\,\mathrm{d}D_{ct} - \left(P_c + q\,\mathrm{sgn}(\mathrm{d}\tilde{\pi}_t)\right)\mathrm{d}\tilde{\pi}_t, \tag{2.8}$$

with $\tilde{\pi}_t$ being the number of shares of the liquid asset c held at time t, P_c being the price of the liquid asset c and P_d being the price of the illiquid asset d.

Again, for ease of exposition, I restrict the model parameters as follows.

Assumption 17. The dynamic of the exposure shocks, as described by (2.4), and the supply S_d of the illiquid asset d satisfy

$$\frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}} \neq S_d \neq \frac{\lambda_{21}}{\lambda_{12}+\lambda_{21}}.$$

Also, the vectors e_d, e_c, and $e_1 - e_2$ are not collinear. □

Assumption (17) prevents the lengthy treatment of non-generic cases.[11]

2.3 Transaction Costs

I investigate how transaction costs on the exchange impact the OTC market.

Three motivations for this extension. First, in the first chapter of this dissertation, I understood a market as being illiquid when an investor cannot immediately trade on it. I then studied the interaction of such an illiquid market with a frictionless market to which investors have a

[10] The cross-sectional distribution of wealth is not necessarily constant over time. However, the equilibrium policies are independent of the wealth, thanks to the CARA preferences, and this non-stationarity has no impact on the equilibrium portfolios and prices.

[11] I endogenize the risk-profile e_c in the first chapter of this dissertation. The assumption regarding the non-collinearity of the vectors of exposures will hold in this case if e_d and $e_1 - e_2$ are not collinear

constant access. For convenience, I thus ignored that trading may very well be executed with barely any delay but at a cost.

Second, on actual markets, there is usually a mix between trading delays and costs of trading. For example, both illiquid bonds and CDS's are traded OTC, usually via dealers. The intermediation of trading by dealers means that trading will involve both delays, because of the time taken to contact the dealer and discuss the details of the transactions, and transaction costs in the form of the fee collected by the dealer. For a trade in the illiquid bond, the intermediary may act as a broker and search for an end-user willing to take the other side of the trade and the main trading friction will presumably be the delay. Quite differently, for a liquid CDS, the dealer is likely to propose a trade immediately and the main friction is likely to be the intermediation fee. In this section, I attempt to capture these two aspects of illiquidity.

Third, Political attempts to tax financial transactions. See French example of a financial transaction tax that applies to CDS trades but not to transactions in debt securities. Appear that relies on assumption that CDS trading makes debt financing more costly.

In this section, I thus explore the interaction between illiquid OTC markets and markets that are liquid, in the sense that a position can be liquidated at any time, but on which trading is costly. My modeling of the transaction costs and the subsequent analysis are based on the section 4 of Gârleanu [2009].

I assume proportional transaction costs for the asset c. Namely, investors can purchase the liquid asset c at the price

$$P_c + q,$$

and sell this same asset at

$$P_c - q.$$

These two prices are taken as given. The constant $q > 0$ measures the bid-ask spread and is exogenous. The mid-price P_c is an equilibrium quantity.

In the static version of the model, an investor only trades on the exchange after being hit by an exposure shock or after trading on the OTC market. This optimal trading activity results from the combination of the CARA preferences with the absence of state variable induces. In particular, the trading on the exchange in the baseline model is fully characterized by the holdings adopted after each of the six possible type shocks. I assume the same type of trading activity in the setting with transaction costs.

Assumption 18. In equilibrium, an investor only trades on the exchange after a change of her type. I denote the portfolios adopted after each possible shock by

shock	portfolio adopted
$1l \to 2l$	$\pi(2l)$
$2l \to 1l$	$\pi(1l_{2l})$
$2l \to 2h$	$\pi(2h_{2l})$
$2h \to 1h$	$\pi(1h)$
$1h \to 2h$	$\pi(2h_{1h})$
$1h \to 1l$	$\pi(1l_{1h})$

and write Π for the vector containing all six portfolios.

A few comments are in order. First, the type $1l$ can be reached from the type $2l$ by an exposure shock and from the type $1h$ by an OTC trade. In the baseline setting, the current type fully determines the holdings in the liquid asset. In the setting with transaction costs, both the fundamental exposure and the trade direction will determine the optimal holdings in the liquid asset. Hence, the optimal portfolio may depend on both the current and last type, something that Assumption 18 allows for. Second, some components of Π may be equal. This happens when the transaction costs are sufficiently large to dominates a potential improvement of the risk-profile. As is intuitively clear, for an arbitrarily large q an investor optimally decide to never trades on the liquid market can be optimal. This corresponds to a vector of holdings in Assumption 18 whose components are all equal. Third, assuming that Assumption 18 does not hold largely amounts to assuming that investors react to the the introduction of the transaction cost by trading more frequently. I do not think that such an assumption can be justified in equilibrium.

An equilibrium with transaction costs is entirely similar to an equilibrium in the baseline model. The only difference is the characterization of the optimal liquid holdings. Given the equilibrium holdings Π and mid-price P_c for the liquid asset and taking as given a trading pattern on the OTC market, the value functions of the agents are still be characterized as the solution to a system of HJB equations. Namely, I assume that the only transaction on the OTC market are the sales of the illiquid asset by $1h$-investors to $2l$-investors. This means that the only type changes induced by meetings on the OTC markets are of the form

$$(1h, 2l) \mapsto (1l, 2h)$$

and, as a result, occur at the rate $\Lambda\mu(1h)\mu(2l)$.[12] Given the trading on the OTC market, I can

[12] In the baseline setting, this trading pattern occurs is the unique equilibrium outcome exactly when

$$\det\left(\begin{pmatrix} a_d & a_c \\ b_d & b_c \end{pmatrix}\right)\det\left(\begin{pmatrix} \alpha_1 - \alpha_2 & a_c \\ \beta_1 - \beta_2 & b_c \end{pmatrix}\right) > 0,$$

as shown in the first chapter of this dissertation. This is still the case for sufficiently small transaction costs q, as write the HJB equations. I then assume the form

$$V(w, i\theta, \pi) = -e^{-r\gamma(w+a(i\theta,\pi)+\bar{a})}$$

for the value functions and follow the same steps as in the first chapter. Again choosing

$$\bar{a} \overset{(\Delta)}{=} \frac{1}{r\gamma}\left(-1+\frac{\rho}{r}+\gamma m_e+\log(r)\right)$$

and recalling the definition

$$\kappa(i\theta,\pi) \overset{(\Delta)}{=} \theta m_d + \pi m_c - \frac{r\gamma}{2}\begin{pmatrix}1 & \theta & \pi\end{pmatrix}\Sigma_i\begin{pmatrix}1\\ \theta\\ \pi\end{pmatrix} \tag{2.9}$$

for the flow of mean-variance benefits, these steps yields the characterization

$$\begin{aligned}
ra(1l_{1h}) &= \kappa(1l,\pi(1l_{1h})) + \mathcal{O}(\gamma)\\
&\quad + \lambda_{12}\left(\begin{array}{l} a(2l)\\ -(\pi(2l)-\pi(1l_{1h}))\left(P_c - q + 2q\mathbb{1}_{\{\pi(2l))>\pi(1l_{1h})\}}\right)\\ -a(1l_{1h})\end{array}\right)\\
ra(1l_{2l}) &= \kappa(1l,(1l_{2l})) + \mathcal{O}(\gamma)\\
&\quad + \lambda_{12}\left(\begin{array}{l} a(2l)\\ (\pi(2l)-\pi(1l_{2l}))\left(P_c - q + 2q\mathbb{1}_{\{\pi(2l))>\pi(1l_{2l})\}}\right)\\ -a(1l_{2l})\end{array}\right)\\
ra(1h) &= \kappa(1h,\pi(1h)) + \mathcal{O}(\gamma)\\
&\quad + \lambda_{12}\left(\begin{array}{l} a(2h_{1h})\\ -(\pi(2h_{1h})-\pi(1h))\left(P_c - q + 2q\mathbb{1}_{\{\pi(2h_{1h})>\pi(1h)\}}\right)\\ -a(1h)\end{array}\right)
\end{aligned}$$

---

shown in Appendix B. For general values of q, I must check the consistency numerically. Note that the proof for small transaction costs in the appendix is not immediate because in the setting with transaction costs 4 trades may, a priori, generate a surplus. These trades are

$$\begin{array}{lcl} (1h,2l) & \mapsto & (1l,2h)\\ (2h,1l) & \mapsto & (2l,1h)\\ \left(1h,1l_{2l}\right) & \mapsto & \left(1l_{1h},1h\right)\\ \left(2h,2l_{1l}\right) & \mapsto & \left(2l_{2h},2h\right)\end{array}.$$

As the two last candidates are original to the setting with transaction costs, a continuity argument based on the baseline setting is not sufficient to characterized when the trade

$$(1h,2l) \mapsto (1l,2h)$$

is the only one that is profitable.

$$
\begin{aligned}
&\quad + 2\lambda\mu(2l)\begin{pmatrix} a(2hs) \\ -(\pi(2hs) - \pi(2l))\left(P_c - q + 2q1_{\{\pi(2hs)>\pi(2l)\}}\right) \\ -\Theta P_d - a(2l) \end{pmatrix} \\
ra(2l) &= \kappa(2l, \pi(2l)) + \mathcal{O}(\gamma) \\
&\quad + \lambda_{21}\begin{pmatrix} a(1l_{2l}) \\ -(\pi(1l_{2l}) - \pi(2l))\left(P_c - q + 2q1_{\{\pi(1l_{2l})>\pi(2l)\}}\right) \\ -a(2l) \end{pmatrix} \\
&\quad + 2\lambda\mu(1h)\begin{pmatrix} a(2h_{2l}) \\ -(\pi(2h_{2l}) - \pi(2l))\left(P_c - q + 2q1_{\{\pi(2h_{2l})>\pi(2l)\}}\right) \\ -\Theta P_d - a(2l) \end{pmatrix} \\
ra(2h_{2l}) &= \kappa(2h, \pi(2h_{2l})) + \mathcal{O}(\gamma) \\
&\quad + \lambda_{21}\begin{pmatrix} a(1h) \\ -(\pi(1h) - \pi(2h_{2l}))\left(P_c - q + 2q1_{\{\pi(1h))>\pi(2h_{2l})\}}\right) \\ -a(1h_{2h}) \end{pmatrix} \\
ra(2h_{1h}) &= \kappa(2h, \pi(2h_{1h})) + \mathcal{O}(\gamma) \\
&\quad + \lambda_{21}\begin{pmatrix} a(1h) \\ -(\pi(1h) - \pi(2h_{1h}))\left(P_{c,0} - q + 2q1_{\{\pi(1h))>\pi(2h_{1h})\}}\right) \\ -a(2h_{1h}) \end{pmatrix}
\end{aligned} \tag{2.10}
$$

of the value functions. The illiquid asset trades at the price

$$P_d = \eta_{2l}(a(1h) - a(1l_{1h})) + \eta_{1h}(a(2h_{2l}) - a(2l)) + \mathcal{O}(\gamma),$$

which is a weighted average of the reservation values of the two traders.

I add two comments. First, the system (2.10) is linear in the unknowns "$a(\cdot)$". As a result, my efforts will be focused on the characterization of the equilibrium holdings Π and mid-price P_c of the liquid asset. The value functions and bargained prices then follow.

Second, there are now six types instead of four in the baseline model. Let us see why. With proportional transaction costs, the marginal cost of buying the liquid asset and the marginal revenues from selling the liquid asset are strictly different. As a result, if the type change

$$1h \mapsto 1l$$

triggers a purchases of the liquid asset but the type change

$$2l \mapsto 1l$$

triggers a sale, the risk profile of an agent of current type $1l$ depends on what her previous types was. This means that a correct analysis of the mode with transaction costs must sometimes complement the current type with the previous type. Formally, I split the type $1l$ into the subtypes $1l_{1h}$ and $1l_{2l}$. Similarly, I split type $2h$ into the subtypes $2h_{2l}$ and $2h_{1h}$. The absence of a well-defined marginal price makes the analysis of the additional sub-types necessary.

The absence of a marginal price also makes a characterization of the equilibrium holdings Π and mid-price P_c challenging and best split into two steps.

The first steps assumes a "pattern" of trading on the liquid market p and derives the optimal holdings $\Pi(p)$. Formally a *pattern of trading on the liquid market* indicates for each type change whether it triggers a purchase, a sale, or no trade on the liquid market. Formally, the pattern of trading p is an element of

$$\mathscr{P} \triangleq \{b, s, \emptyset\}^{\#\{\text{type shocks}\}},$$

with b standing for "buy", s for "sell", $\emptyset$ for "do not trade", and $\#\{\text{type shocks}\}$ being the number of different type shocks that an investor can incur. As can be checked, in Figure 2.4, this number equals 6 in the current setting.

If the investor places, say, a buy order, both an adjustment upward and an adjustment downward of the order are done with the ask-price as the marginal price. In particular, by defining when investors trade and in which direction, I defined a set of marginal prices from which I can derive optimal holdings. This is the object of the next proposition.

Proposition 19. *Let us take as given a pattern p of trading on the liquid market. Then, the equilibrium mid-price $P_c(p)$ and holdings $\Pi(p)$ for the liquid asset are uniquely characterized as follows.*

1. *For any time t, the holdings process $(\pi_t)_t$ satisfies both*

$$\begin{aligned} P_c + q \geq \mathrm{E}_t &\left[\int_t^\tau e^{-r(s-t)} \left(\partial_\pi \kappa \left(i_s \theta_s, \pi_t\right)\right) \mathrm{d}s\right] \\ &+ \mathrm{E}_t\left[e^{-r\tau}\left(P_c + q \mathrm{P}_t\left[\textit{buy next}\right] - q \mathrm{P}_t\left[\textit{sell next}\right]\right)\right] + \mathscr{O}(\gamma) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} P_c - q \leq \mathrm{E}_t &\left[\int_t^\tau e^{-r(s-t)} \left(\partial_\pi \kappa \left(i_s \theta_s, \pi_t\right)\right) \mathrm{d}s\right] \\ &+ \mathrm{E}_t\left[e^{-r\tau}\left(P_c + q \mathrm{P}_t\left[\textit{buy next}\right] - q \mathrm{P}_t\left[\textit{sell next}\right]\right)\right] + \mathscr{O}(\gamma), \end{aligned} \tag{2.12}$$

with

$$\tau \triangleq \inf\{s > t : \pi_s \neq \pi_{s-}\}$$

being the time of the next trade on the liquid market,

$$\mathrm{P}_t\left[\textit{buy next}\right] \stackrel{\Delta}{=} \mathrm{P}_t\left[\pi_\tau > \pi_{\tau-}\right]$$

being the probability of the next trade being a purchase, and

$$\mathrm{P}_t\left[\textit{sell next}\right] \stackrel{\Delta}{=} \mathrm{P}_t\left[\pi_\tau < \pi_{\tau-}\right]$$

being the probability of the next trade being a sale.

2. *If τ coincides with the next change of type, the first-order conditions* (2.11) *and* (2.12) *become*

$$P_c \geq \frac{1}{r}\left(\partial_\pi \kappa\left(i_t \theta_t, \pi_t\right)\right) - q - 2q\,\mathrm{P}_t\left[\pi_\tau > \pi_{\tau-}\right]\frac{\lambda_\tau}{r} \tag{2.13}$$

and

$$P_c \leq \frac{1}{r}\left(\partial_\pi \kappa\left(i_t \theta_t, \pi_t\right)\right) + q + 2q\,\mathrm{P}_t\left[\pi_\tau > \pi_{\tau-}\right]\frac{\lambda_\tau}{r}, \tag{2.14}$$

respectively, with τ being the arrival rate of the next type change.

3. *If the investor purchased the liquid asset at time t, meaning that*

$$\pi_t > \pi_{t-},$$

then, the inequality (2.11) *holds with equality. Similarly, if the investors sold the liquid asset at time t, meaning that*

$$\pi_t < \pi_{t-},$$

then the inequality (2.12) *holds with equality.*

4. *The optimal holdings adopted after a trade are affine decreasing in the price P_c. In particular, there is exactly one equilibrium price P_c that clears the liquid market.*

Proof. See the Appendix. The argument is based on Section 4.2 in Gârleanu [2009]. □

If all the components of $\hat{\pi}$ are distinct, the time of the next trade always coincides with the next type change and we can focus on the case 2 of the last proposition. However, when the transaction costs are relatively large, it may be optimal to sometimes refrain from trading after a type change. This will translate into some of the components of Π being identical, and into the next trade time not being distributed as an exponential random variable. As a result, we must sometimes rely on the general first-order conditions in the point 1 of the last proposition.

I now turn to the second steps for the characterization of the equilibrium mid-price P_c and holdings Π. This second step starts with the vector of holdings defined in Proposition 19 and derives a new pattern p' for the liquid market. Namely, the differences between the holdings in Proposition 19 define when the investors buy and when they sell. This defines the components of p' corresponding to the values "buy" or "sell" in the original pattern p. For those type changes that did not trigger any trading in p, we can check whether this behavior is optimal. Namely, we can check whether the first-order condition for a marginal purchase (2.11) holds and, should this not be the case, set the relevant component to p' equal to "buy". Similarly, should (2.12) not hold, we set the relevant component of p' equal to "sell".

Combining the two steps defines a mapping from the set $\mathscr{P}$ of trading patterns into itself $\mathscr{P}$ and solving for the equilibrium mid-price P_c and holdings Π amounts to finding a fixed-point of this mapping. The next proposition indicates that such a fixed-point always exists. The proof of the proposition is constructive and the basis for a numerical solution of the model.

Proposition 20. *For any transaction costs q, there exists an equilibrium with transaction costs.*[13] *These equilibria are continuous in q and are characterized by a sequence of thresholds*

$$\hat{q}_1, \hat{q}_2, \ldots \hat{q}_5.$$

As long as the transaction costs are small enough, meaning

$$q < \hat{q}_1,$$

then, the pattern of trading $p \in \mathscr{P}$ on the liquid market is the same as in the baseline setting without transaction costs. For larger transaction costs, meaning

$$\hat{q}_1 \leq q < \hat{q}_2,$$

one of the transactions in the pattern p collapses, meaning that one of the components of p switches from either "buy" or "sell" to "do not trade". One more transactions collapses at each thresholds. For large enough transaction costs, meaning

$$\hat{q}_5 \leq q,$$

there is no trading at all on the liquid market and each investor holds the same holdings S_c in the liquid asset c.[14]

[13] Recall, however, that the ex ante assumption regarding the trading pattern on the OTC market must hold.

[14] Arguably, referring to an asset that nobody trades as being liquid may seem clumsy. I however understand an

The equilibrium holdings Π *and price* P_c*, along with the thresholds* $\hat{q}_1 \dots \hat{q}_5$*, can be characterized in closed-form.*

Proof. See the appendix. □

The exact equilibrium behavior depends on the relative values of the exogenous parameters. In particular, an exhaustive description of the order in which the six trades collapse requires the handling of a large number cases.[15] These cases are qualitatively similar but distinct in terms of algebraic expressions. For this reason, I make certain parametric assumptions.

First, I want the assets to be *substitutes* for each other, meaning that investors buy the liquid asset while attempting to buy on the OTC market and then reduce their exposure to the liquid asset once the trade on the OTC market is completed. As shown below, the suitable assumptions to capture this behavior are

$$\sigma_{cd} \overset{(\Delta)}{=} \begin{pmatrix} a_c \\ b_c \end{pmatrix} \cdot \begin{pmatrix} a_d \\ b_d \end{pmatrix} > 0 \tag{2.15}$$

and

$$\begin{pmatrix} a_c \\ b_c \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 - \alpha_2 \\ \beta_1 - \beta_2 \end{pmatrix} > 0. \tag{2.16}$$

Second, I want the marginal buyer of the asset to have a high valuation. A marginal investor with a high valuation for the asset ensures that illiquidity decreases the price of the illiquid asset. Keeping in mind our prime example of the OTC market as being a bond market, it seems empirically clear that the illiquidity spread should be positive. Third, I assume that a majority of investors have a high valuation for the illiquid asset. Equivalently, I may assume that a given investors spends more time with a high valuation than with a low valuation. This is consistent with an interpretation of the low-valuation state representing a transitory crisis requiring the liquidation of certain illiquid positions. The next assumption collects these three requirements on the model parameters.

asset as being liquid if it can be traded without delay. Even with large transaction costs, investors can trade the liquid asset whenever they want.

[15] In the baseline model, the trading on the liquid market can be decomposed as a component due to a trade on the OTC market and a component due to a change of exposure. Depending on the sign of each component, there can be $4 = 2 \times 2$ asymptotic trading patterns p when the transaction costs q vanish. Each asymptotic trading pattern comprises 6 trades. A priori, there are $5! = 120$ ways of choosing the order in which these trades vanish (the two last trades must vanish simultaneously). A gross a priori estimate of the number of cases to consider for a comprehensive analytic characterization of an equilibrium with transaction costs is thus $4 \times 5! = 480$. This number must be adjusted downward, because certain trades must disappear before other ones, as can be seen in the proof. The overall number of cases would remain substantial, though.

Assumption 21. 1. The liquid asset is a substitute for the illiquid one, meaning

$$\begin{aligned} \sigma_{cd} &\overset{(\Delta)}{=} \begin{pmatrix} a_c \\ b_c \end{pmatrix} \cdot \begin{pmatrix} a_d \\ b_d \end{pmatrix} > 0, \\ \begin{pmatrix} a_c \\ b_c \end{pmatrix} &\cdot \begin{pmatrix} \alpha_1 - \alpha_2 \\ \beta_1 - \beta_2 \end{pmatrix} > 0. \end{aligned} \tag{2.17}$$

2. A spell with a low valuation is shorter on average than a spell with a high valuation, meaning

$$\lambda_{12} > \lambda_{21}.$$

3. The marginal holder of the asset traded OTC has a high valuation, meaning

$$\mu_{2\bullet} > \frac{S_d}{\Theta}.$$

In the next proposition, I partially characterize an equilibrium under Assumption 21.

Proposition 22. *Let us assume that Assumptions 21 holds. The threshold $\hat{q}_1$ until which every type change triggers a trade is*

$$\hat{q}_1 = \frac{1}{2} \frac{r\gamma\Theta\sigma_{cd}}{r + \lambda_{12} + 2\Lambda\mu(2l)}.$$

For $q < \hat{q}_1$, the trading pattern is

$$\text{investors} \left\{ \begin{array}{l|l} \textit{whose valuation for the illiquid asset increases} & \textit{buy} \\ \textit{whose valuation for the illiquid asset decreases} & \textit{sell} \\ \textit{who buy the illiquid asset} & \textit{sell} \\ \textit{who sell the illiquid asset} & \textit{buy} \end{array} \right\} \text{the liquid asset.}$$

and the equilibrium mid-price P_c is

$$P_c = P_c|_{q=0} + \Phi_0 q + \mathcal{O}(\gamma), \tag{2.18}$$

with the limit value $P_c|_{q=0}$ being the same as the price in the baseline setting and the sensitivity Φ_0 being

$$\Phi_0 \overset{\Delta}{=} \left(\frac{1}{\lambda_{12}} - \frac{1}{\lambda_{21}} \right) \left(\lambda_{21}\mu(2l) + \lambda_{12}\mu(1h) - 2\Lambda\mu(1h)\mu(2l) \right).$$

The sensitivity Φ_0 is increasing in the meeting rate λ.

For larger transaction costs, meaning

$$\hat{q}_1 < q,$$

the sellers of the illiquid asset keep the same liquid holdings both while searching for a counter-party and after having traded OTC. For transaction costs in the interval

$$q \in \left[\hat{q}_1, \hat{q}_2\right),$$

the equilibrium mid-price P_c is

$$P_c = P_c|_{q=0} + \Phi_1 q + \mathcal{O}(\gamma), \tag{2.19}$$

with

$$\Phi_1 \stackrel{\Delta}{=} \Phi_0 + \frac{2}{\lambda_{12}} 2\Lambda\mu(1h)\mu(2l).$$

An explicit expression for $\hat{q}_2$ is stated in the proof.

The sellers are on the short-side of the market. Finding a counter-party is thus comparatively easy for them and they expect the search for a counter-party to be brief. When weighing the costs and benefits of hedging a sub-optimal on the liquid market, the sellers are thus the first ones to reduce their trading frequency.

The mid-price P_c is increasing in the meeting rate Λ. The intuition for this result is the following. First, by increasing the meeting rate Λ, the volume of trading on the OTC market increases, meaning that the rate at which investors undo their hedges increases as well. Hence increasing Λ increases the rate at which investors enter both the type

$$1l_{1h}$$

of those agents having successfully sold the illiquid asset and the type

$$2h_{2l}$$

of those agents having successfully bought the illiquid asset. Now, Assumption 21 ensures that investors recovers from the low-valuation state $1l_{1h}$ faster than they drop from the high-valuation state $2h_{2l}$. Hence, in a stationary equilibrium, it must be that the mass

$$\mu(1l_{1h})$$

increases by less than

$$\mu(2h_{2l}).$$

As the $1l_{1h}$ investors last bought the liquid asset and the $2h_{2l}$ investors last sold the liquid asset, increasing the meeting rate Λ increases the proportion of investors across the population who last sold the liquid asset. Proposition 19 shows how this increase increases the demand for the liquid asset and, as a result, increases the equilibrium price P_c.

The mid-price P_c is increasing in Λ. Importantly, however, the joint effect of the search friction and the transaction costs on P_c is ambiguous. More specifically, the price difference

$$P_c - \left(P_c|_{(q,\frac{1}{\Lambda})=(0,0)} \right)$$

can be either positive or negative, with an exact characterization depending on the difference

$$\mu_{2\bullet} - \frac{S_d}{\Theta}$$

between the proportion of investors with a high-valuation and the supply of the illiquid asset.

When the first trade collapses at $q = \hat{q}_1$, the dependence of the mid-price on the transaction costs is shifted by

$$\frac{2}{\lambda_{12}} 2\Lambda\mu(1h)\mu(2l),$$

a positive quantity. This is intuitive. Indeed, the first trade to collapse is a purchase. Hence, this collapse increases the proportion of investors who last sold the liquid asset. According to Proposition 19, this increases the demand for the liquid asset, which results in a higher equilibrium price P_c.

To sum up, propositions 20 and 22 describe how trading on the liquid market collapses when the transaction costs increase. The most interesting consequences of this collapse is probably its impact on the OTC market.

The solution to the HJB equations (2.10) can be characterized in closed-form using an appropriate software. I can thus explore how the transaction costs on the liquid market impact the price of the asset traded OTC. I do so in Figure 2.6.[16]

[16] The equilibrium quantities are, strictly speaking, available in closed-form. However, the expressions for the equilibrium quantities are cumbersome and, in my view, difficult to interpret. This is why I focus on a numerical example. Further, checking the consistency of the trading pattern on the OTC market is also a cumbersome exercise that is best performed numerically.

Figures 2.5 shows us an ambiguous relationship between the transaction costs incurred on the exchange and the prices bargained on the OTC market. In particular, the bargained prices are first decreasing in the transaction costs, up to a threshold, and then tends to be increasing. The threshold is increasing in the search friction on the OTC market.

This pattern can be intuitively understood with the help of Proposition 22. The first thing to realize is that investors are more affected by the transaction costs if they trade more. Now, investors on the short-side of market trade faster on OTC market because they hedge and unhedge at a higher rate on the liquid market. As a result, they are comparatively more affected by an increase of the transaction costs. Following the literature, I chose the seller of the illiquid asset to be on the short side of the market. As a result, at first, any increase of the transaction costs is relatively more harmful to the sellers' outside options, which decrease the bargained price. This explains the decreasing portion of the curves in Figure 2.5. If the transaction costs increases further, investors must balance the cost and benefits of hedging a sub-optimal on the OTC market with the liquid asset. On the one hand, hedging the sub-optimal exposure improves the risk-profile while searching for a counter-party. On the other hand hedging and undoing the hedge means that round-trip transaction costs are incurred. Intuitively, the hedge is only rational if the expected search time is large enough when compared with the transaction costs. The agents on the short-side of the market can trade faster on the OTC market. As a result, when the transaction costs q increase, they are the first ones to refrain from hedging with the liquid asset. The point at which the investors on the short side of the market adjust their trading pattern for the liquid asset is the threshold $\hat{q}$ in Proposition 22. This threshold corresponds to the kinks in Figure 2.5. Beyond the threshold $\hat{q}$, investors on the long side of the market trade more on average that investors on the short side. As a result, any further increase of the transaction costs will disproportionately hurt those agents on the long side of the market. As the buyers are on the long side of the market, any such increase of the transaction costs makes the bargaining position of the buyers relatively worse, which induces a price increase.

Finally, interesting to note that the spillover effect of the transaction costs on the OTC market is much stronger than the effect on the liquid market itself. The spillover effect is of the order of 50bp for transactin costs of 50bp, whereas the effect on the liquid market itself is approximately five times smaller.

2.4 Conclusion

I studied an economy in which risk-averse investors share endowment risk by trading two imperfectly correlated assets. The first one is traded on an exchange, with a constant bid-ask spread, whereas the second is traded on on illiquid OTC market. I follow Duffie et al. [2007] for the modeling of the OTC market. Liquidity, measured by the meeting rate, affects investors both because contacting a potential trading partner is time consuming and because prices are not competitive. The interaction between the two markets is driven by the investors who use the exchange as a substitute for the OTC market and hedge on the exchange while searching for a counter-party on the OTC market.

I show the existence of an equilibrium for any level of transaction costs and obtain closed-form expressions for the equilibrium quantities. The existence argument is non standard because the transaction costs introduce discontinuities in the equilibrium equations.

The search frictions on the OTC market affect the price of the asset on the exchange. As made clear by Gârleanu [2009], the equilibrium effect of transaction costs depends on the imbalance between the investors who last purchased and those who last sold the asset. When the meeting rate on the OTC market varies, so does the rate at which investors hedge (by both buying and selling) on the exchange, and whether the buying or selling pressures dominates depends on the dynamics of the hedging demand. I show that, when low valuation spells are relatively short, the search friction on the OTC market increase the risk premium on the exchange.

The transaction costs on the exchange have two main effects. First, they make the marginal cost of trading on the exchange higher. Investors optimally react to these costs by reducing the size of their transactions *when they trade*. Second, the transaction costs also make trading on the exchange less frequent. Indeed, when an investors receives a shock to her hedging demand, she balances the costs of trading on the exchange, the optimal risk-profile that she may achieve by trading, and when this risk-profile will have to be adjusted again. When the transaction costs are high enough, the round-trip transaction costs can dwarf the improvement to the risk-exposure and investors optimally decide not to trade after certain idiosyncratic shocks.

As the transaction costs increase the costs of hedging, they also affect the outside options of the agents bargaining on the OTC market. As I show with numerical examples, the relationship between the transaction costs on the exchange and the price bargained on the OTC market is non-monotonic.

I can draw a parallel between the predictions of my model and current regulatory proposals aiming at taxing CDS trading. The equilibrium behavior of my model indicates that making CDS trading costly may indeed decrease bond yields. The equilibrium relationship between taxes and yields is, however, non-monotonic, and depends in a rather subtle way on the strength of the trade motives, their dynamics, and the relative levels of the two types of illiquidity. A financial transaction tax may thus decrease debt financing, but the exact effect is hard to predict. Introducing a financial transaction tax to decrease borrowing yields seems to be a hazardous strategy.

In some future work, I intend to intermediate the level of frictions on the two markets. More specifically, the level of transaction costs on the exchange will be optimally chosen by an intermediary maximizing her revenues, whereas the trading on the OTC market will be intermediated by dealers who optimally choose, for a cost, the frequency at which they can get in touch with investors.

2.5 Figures

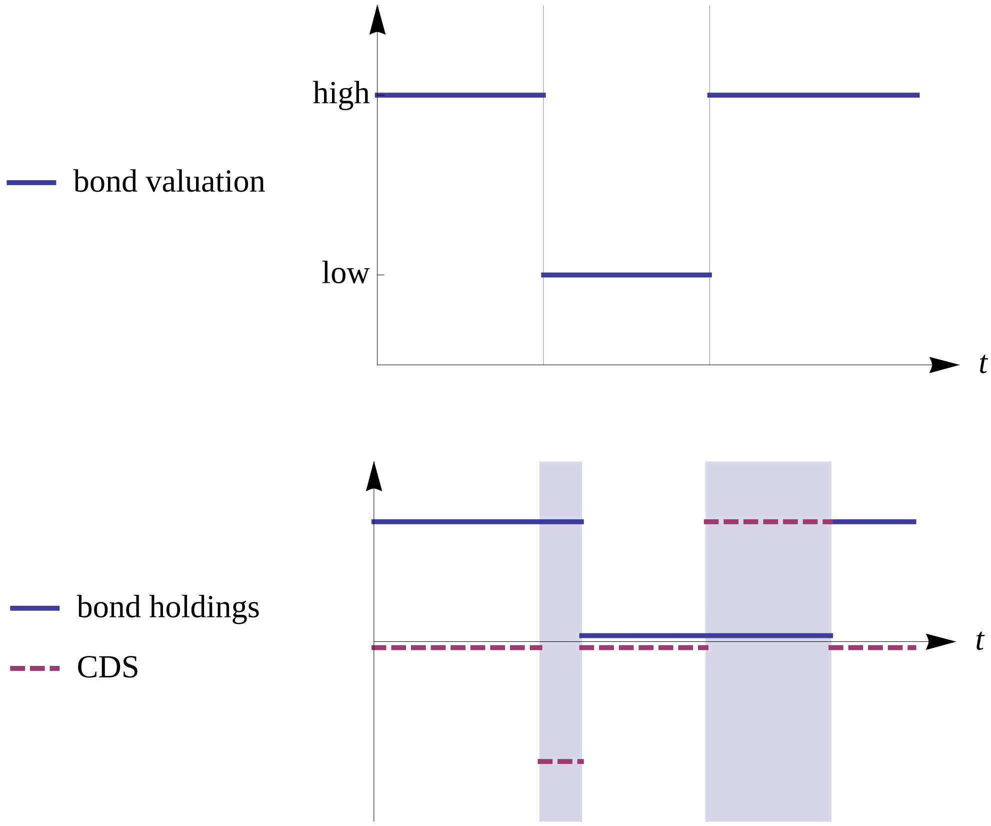

Figure 2.1: These plots represent stylized paths for the valuation (above) and holdings (below) of a given investor. For the holdings, the solid line represents the holdings in the illiquid asset traded OTC, say a bond, and the dashed line the holdings in the liquid asset, say a CDS contract (as a protection seller). The shaded areas represent the periods during which the investor is searching for a counter-party on the OTC market to re-balance her illiquid holdings. During these periods, the exposure to the asset traded on the exchange is adjusted so as to mitigate the sub-optimal exposure to the asset traded OTC. These plots illustrative and not based on the parameters in Table 1.1

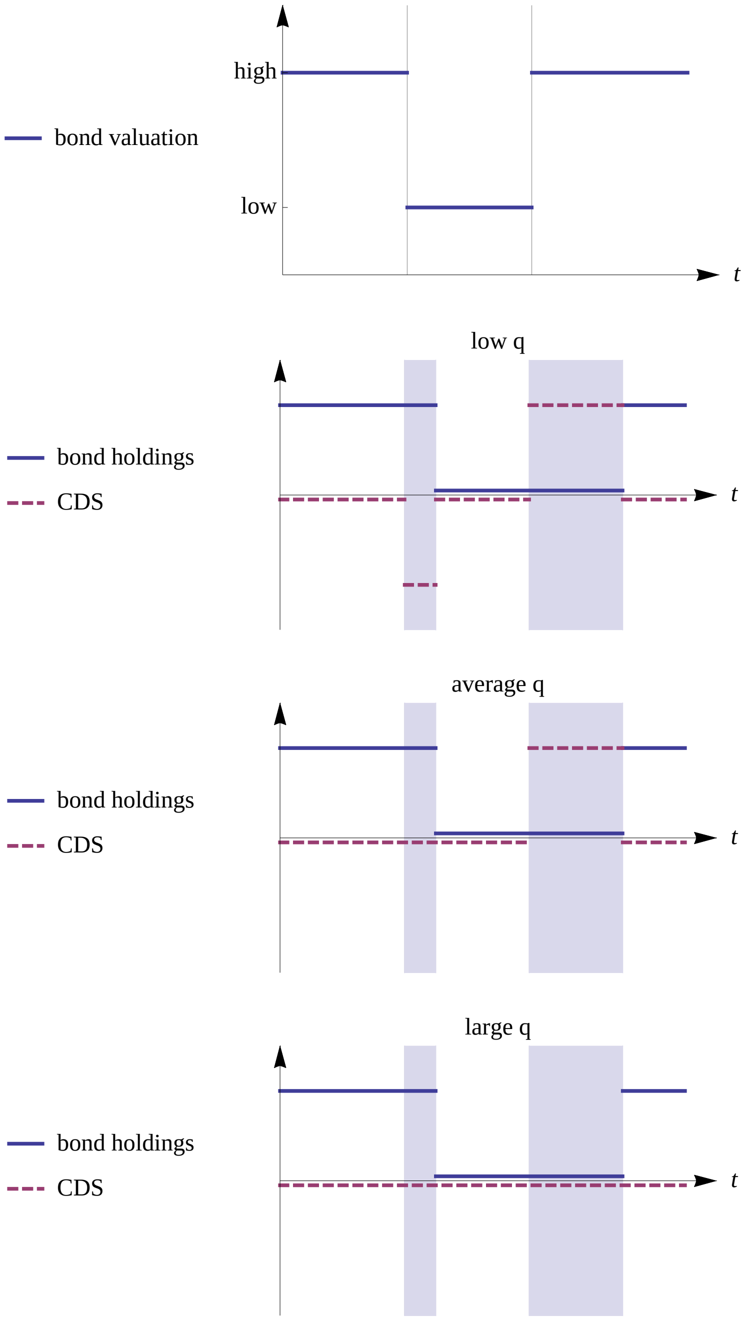

Figure 2.2: These plots illustrate the impact of introducing transaction costs on the liquid market. The path of valuations, at the top, is the same as in Figure 2.1. There are then three stylized paths of holdings in the illiquid asset traded OTC, say a bond, and the holdings in the liquid asset, say a CDS contract (as a protection seller). In the second panel, the transaction costs q are low (second panel) and the investor adjusts her risk-profile using the CDS whenever she is trying to trade the bond. In the third panel, the transaction costs q are larger and an investor only find it optimal to trade the CDS, and incur the round-trip transaction costs, when she expects the re-balancing of her bond portfolio to be a lengthy process. For example, the investor hedges her exposure when she is on the long side of the market but not when she is on the short side. In the fourth panel, the transaction costs q are so high that investors do not hedge their sub-optimal bond exposure anymore. These plots are illustrative and not based on the parameters in Table 1.1.

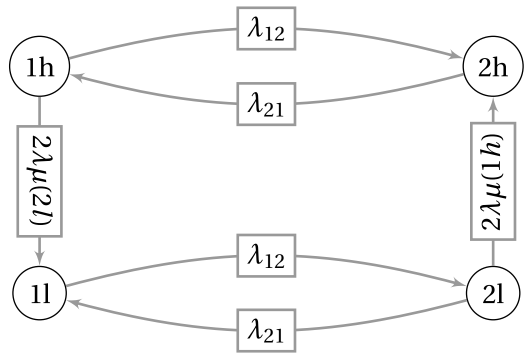

Figure 2.3: The four types of investors in the base model. Each arrow indicates a flow. The number on each arrow indicates the transition intensity for a given investor.

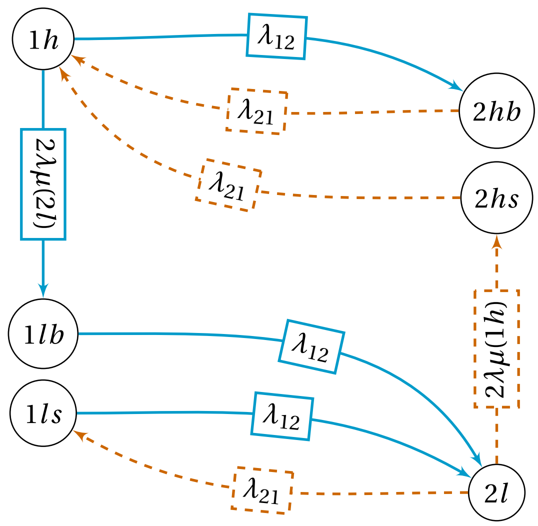

Figure 2.4: The six types of investors in the model with proportional transaction costs. Each arrow indicates a type flow. The number on each arrow indicates the transition intensity for a given investor. A solid arrow indicates a buy transaction on the exchange. A dashed arrow indicates a sell transaction on the exchange. The transaction signs are based on the assumption that 2-agents have a high valuation of the asset traded OTC.

Table 2.1: Base parameters for the setting with transaction costs

notation	parameter	base value
S_c	supply of the asset on the exchange	1
S_d	supply of the asset traded OTC	1
η_{1h}, η_{2h}	bargaining powers	$\frac{1}{2}$
λ_{21}	arrival rate of idiosyncratic liquidity shocks	$\frac{1}{5}$
λ_{12}	recovery rate from a liquidity shocks	2
r	risk-free rate	0.037
m_c	expected payouts of the asset on the exchange	0.05
m_d	expected payouts of the asset traded OTC	0.05
(a_d, b_d)	exposures of asset traded OTC	$(-0.076, -0.528)$
(a_c, b_c)	exposures of liquid asset	$(-0.534, 0)$
(α_1, β_1)	exposures of endowment of type 1	$(-1.335, 1.156)$
(α_2, β_2)	exposures of endowment of type 1	$(0.133, -0.116)$
γ	coefficient of absolute risk aversion	5.01
Θ	holdings in the asset traded OTC	2.5

Choice of the parameters The asset supplies are normalized to 1 and I assume symmetric bargaining powers. The dynamics of the idiosyncratic shocks are taken from Duffie et al. [2007]. The risk-free rate and expected payouts of the assets are the same as in Gârleanu [2009] (the calibration in Gârleanu [2009] is itself based on Campbell and Kyle [1993] and Lo et al. [2004]). The exposures of the assets and endowments are chosen to satisfy the three following conditions.

1. The payouts of each risky asset have a volatility of 0.285, which is the same as in Gârleanu [2009].
2. The correlation between the payouts of the two risky asset is 0.5, which is the same as in Longstaff [2009].
3. There is no aggregate endowment risk, which is the same as in Gârleanu [2009].

These three conditions do not uniquely characterize the exposures. As a result, I arbitrarily set b_c to zero. This leaves four possible sets of parameters, among which I again choose arbitrarily. Finally, the risk aversion γ and holdings Θ in the illiquid asset are chosen to have an equity premium return of 0.031 for each or the risky asset when the search friction vanishes. This equity premium is the same as in Gârleanu [2009].

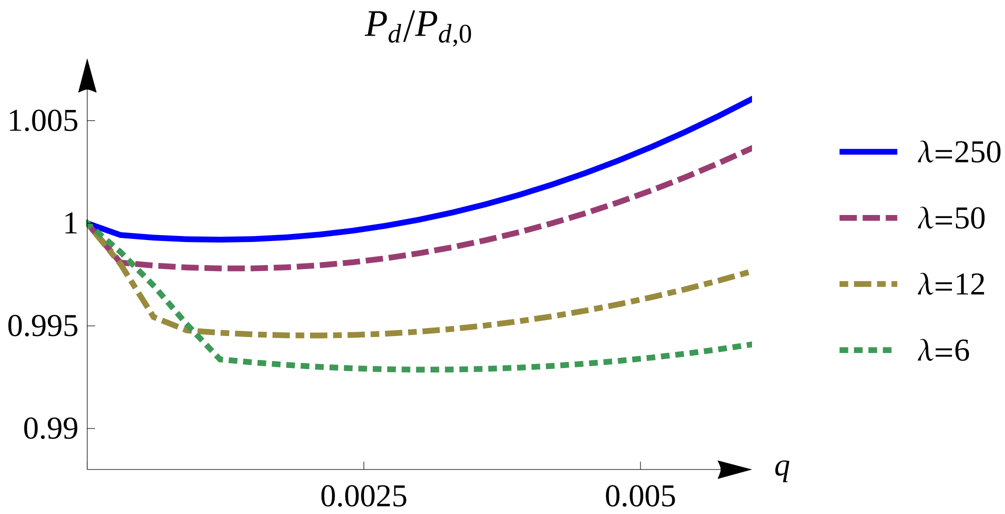

Figure 2.5: Price P_d of the illiquid asset traded OTC as a function of the transaction costs q for the liquid asset. For the sake of comparison, the price P_d is normalized by the value $P_{d,0}$ corresponding to the absence of transaction costs. The relationship between the transaction costs on the liquid market and the price of the illiquid asset traded OTC is ambiguous. When the transaction costs are small when compared to the search friction, the price bargained OTC is decreasing in the transaction costs. Beyond a certain threshold, however, the price bargained OTC becomes increasing in the transaction costs. This figure 2.5 is based on the parametrization in Table 1.1.

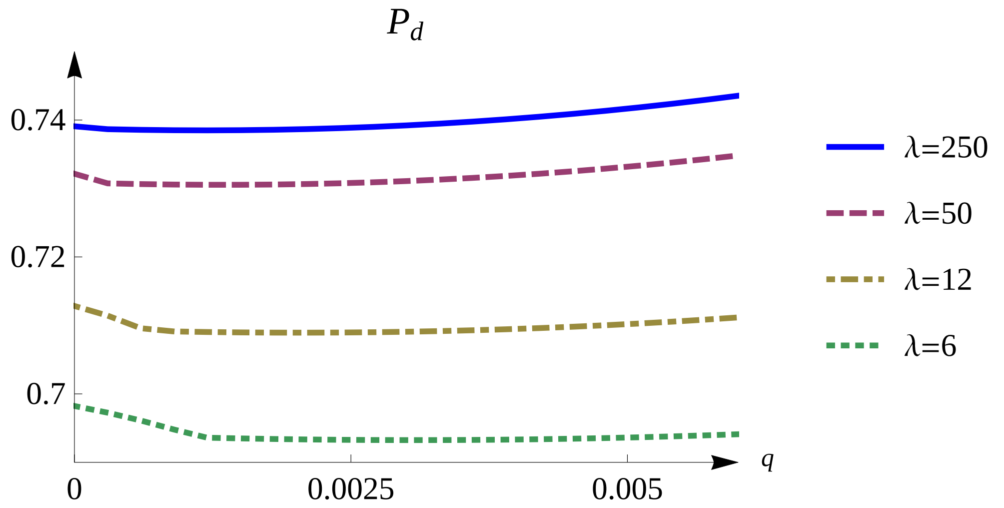

Figure 2.6: Price P_d of the illiquid asset traded OTC as a function of the transaction costs q for the liquid asset. This plot displays the absolute prices used to calculate the relative prices in Figure 2.5. The price effect of the search friction on the OTC market appears to dominate the spill-over effect coming from the transaction costs on the liquid market. This figure 2.6 is based on the parametrization in Table 1.1.

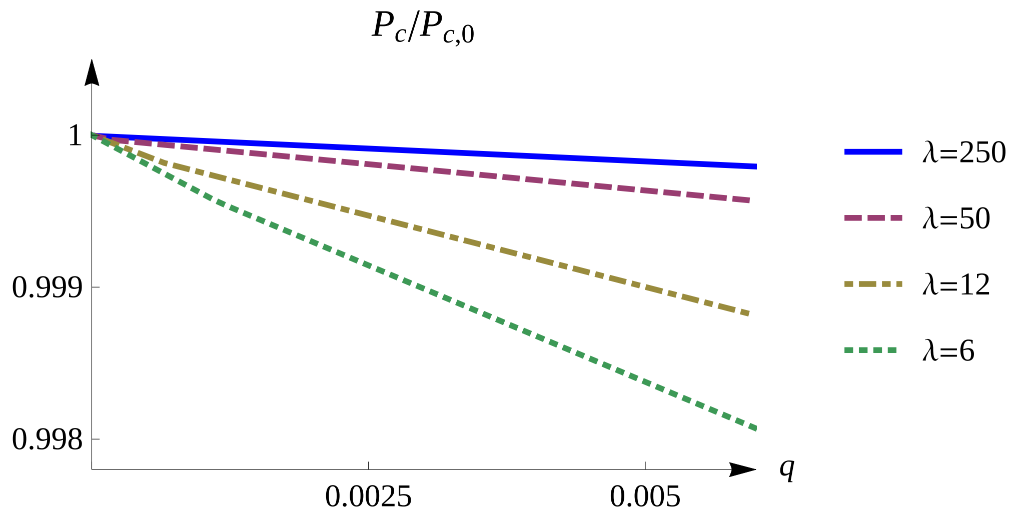

Figure 2.7: Price P_c of the liquid asset as a function of its transaction costs q. The price P_c of the liquid asset is decreasing in the transaction costs. The sensitivity of P_c to the transaction costs is increasing in the search friction. In terms of magnitude, the impact of the transaction costs q on the price P_c of the liquid asset is of the order of 10bp. This is about one fifth of the spillover effect induced by the transaction costs on the OTC market. The magnitude of the spillover effect is of about 50bp, as shown in Figure 2.5. This figure 2.7 is based on the parametrization in Table 1.1.

3 Asymmetric Information and Inventory Concerns in Over-the-Counter Markets

We[1] study how transparency, modeled as information about one's counterparty liquidity needs, affects the functioning of an over-the-counter market. In our model, investors hedge endowment risk by trading bilaterally in a search-and-matching environment. We construct a bargaining procedure that accommodates information asymmetry regarding investors' inventories. Both the trade size and the trade price are endogenously determined. Increased transparency improves the allocative efficiency of the market. However, it simultaneously increases inventory costs, and leads to a higher cross-sectional dispersion of transaction prices. For investors with large risk exposure, the increase of the inventory costs dominates the benefits of the market efficiency. We link the model's predictions to recent empirical findings regarding the effect of the TRACE reporting system on bond market liquidity.

3.1 Introduction

A common concern about over-the-counter (OTC) markets is their opaqueness—investors transact, often unaware of prices available from other counterparties and with little knowledge of trades negotiated recently.[2] Given the important role that OTC markets played in the global financial crisis, many regulators have attempted to shed some light on those so-called dark markets.[3] Perhaps the most notable reform aiming at an increased transparency was the

[1]This chapter is joint work with Julien Cujean (Robert H. Smith School of Business, University of Maryland, 4466 Van Munching Hall, College Park, MD 20742, USA; `+1 (301) 405 7707`; `www.juliencujean.com`; `jcujean@rhsmith.umd.edu`). We thank Pierre Collin-Dufresne, Jens Dick-Nielsen, Julien Hugonnier, Semyon Malamud, and Lasse Pedersen for their comments and suggestions.

[2]For a reference, see Duffie [2012].

[3]For example, the Financial Stability Board names transparency and the public dissemination of trade data as a main objective of introducing trade repositories for OTC derivatives trading. See, for instance, Board [2013] for more detail.

Dodd-Frank Act.[4]

There are costs and benefits associated with an increased transparency. For instance, post-trade transparency—the availability of past transaction data—may lead to a more efficient asset allocation. It may, however, expose dealers to predatory behaviors and significantly reduce their incentives to take on inventory risk and provide liquidity. It is thus important question to determine to whom the costs and benefits of transparency accrue.

To address this question, we develop a general equilibrium model of an OTC market in which investors trade an asset to share endowment risk. Trading the asset first requires finding a counter-party, and we follow the search-and-matching approach developed by Duffie, Garleanu, and Pedersen (2005, 2007). Upon matching, agents bargain on the conditions of the transaction. To model transparency, we introduce information asymmetry among traders. We define *transparency* as a traders' ability to get information about their counterparties' inventory. Information asymmetry creates an adverse selection problem that makes it more difficult to execute large trades.[5] This aspect allows us to determine how regulatory requirements (e.g., TRACE) make inventories riskier.

In the presence of asymmetric information, the usual Nash bargaining solution characterizing bilateral trades is inadequate. We select an alternative bargaining protocol, which resembles the real-world mechanism used in the dealership market. Namely, one agent (say agent 1) posts a quote and the other (say agent 2) decides how many shares to buy or sell at that price.[6] We assume that, before posting the quote, agent 1 receives a signal about the inventory of agent 2. We view this signal as an attempt of agent 1 to extract information about agent 2's liquidity needs from past trading data, a natural outcome of post-trade transparency. The quality of this signal precisely captures the notion of transparency we pursue: the more detailed trading data is available, the better the information about inventory concerns of any given trader is.[7]

Information asymmetry significantly complicates the analysis of the model. Our model, however, remains very tractable. We first solve for an investor's optimal trading strategies taking the cross-sectional distribution of inventories as given. Second, we endogenize this cross-sectional distribution. In particular, optimal trading strategies define the inventory dynamics, which, in turn, determine the cross-sectional distribution of inventories. We demonstrate that optimal strategies are linear in the inventory and signal of investors, making

[4] The corresponding regulation for the US Bond market, the *Trade Reporting And Compliance Engine* (TRACE) exists since 2002. The corresponding European reform is the Market in Financial Instruments Directive (MiFID II).

[5] A quote request is usually followed by both a bid and an ask price being quoted. In this case, both prices would probably be tilted to disguise the attempted rip-off.

[6] Quotes on bond markets do not usually depend on the quantity exchanged. See, Li and Schüerhoff (2012).

[7] The effects of post-trade transparency on inventory risk are particularly strong is markets with moderate/slow trading activity. In this case, even anonymized post-trade transparency can make it possible to infer traders' identities from post-trade data.

it possible to solve for a stationary equilibrium in closed form.

We show that an increase in transparency has three main implications for inventory costs and several dimensions of liquidity.[8] First, we show that transparency always increases inventory costs. This happens via two different channels. On the one hand, transparency exposes any given investor with large inventories to predatory pricing. On the other hand, as transparency improves the allocative efficiency of the market, it becomes more difficult for an investor with excessive exposure to find a counter-party with large and opposite liquidity needs. This second effect exacerbates the first one and is driven by the endogenous distribution of inventories.

Second, we get the more intuitive result that increased transparency always leads to a more efficient allocation of the asset, leading to less dispersion in inventory risk across the population of investors. As a consequence, we show that the cross-sectional variance of the trade sizes at any given moment is monotone decreasing in the degree of transparency of the OTC market.[9]

These two implications together imply that the effect of transparency on investors' value function is ambiguous. On average, investors benefit from an increased transparency and the resulting improvement in the allocative efficiency. In particular, transparency improves welfare. However, those investors with a sufficiently large (long or short) exposure find it increasingly costly to liquidate their position and would benefit from a more opaque market. We obtain an explicit expression for the exposure levels starting from which investors prefer opacity to transparency. This result is in line with the heterogeneous reactions to the introduction of TRACE, with negative reactions on the part of many institutional investors.[10] As our model predicts, transparency is detrimental to agents facing large exposures.

Finally, we show that the price dispersion—the cross-sectional variance of the transaction prices at a given moment—on the OTC market is increasing in the transparency of that market. Two opposite channels operate to generate this result. First, investors being risk averse, the price of a transaction depends on investors' reservation values for trading. As transparency tends to make risk sharing more efficient and to reduce the dispersion of the reservation values, there is a first channel whereby transparency reduces the cross-sectional dispersion of prices. Second, transparency simultaneously increases inventory costs, and the price of a given transaction will therefore drift away from the competitive price when transparency increases. In equilibrium, this second effect dominates, and the dispersion of prices increases with the transparency of the market.

[8] We define the inventory costs as the reservation value of an investor who deviates from a zero risk exposure.

[9] A more natural measure of the trade sizes would be the average size. This is, actually, the average across all the trades of the *absolute value* of the quantity exchanged. Due to technical difficulties, we cannot characterize this quantity analytically.

[10] See, for instance, Decker [2007] and Bessembinder and Maxwell [2008].

Literature review Our model builds on the literature modeling OTC markets. This literature started, to a large extent, with Duffie et al. [2005] and Duffie et al. [2007]. The bilateral trades in these models are characterized by the Nash bargaining solution and, as a result, not naturally suited to accommodate information asymmetry, inter-dealer market but no inventories. Furthermore, the transaction size is exogenously fixed, which prevents a discussion of the different costs and benefits to agents with moderate and large liquidity needs.

An alternative strand of literature considers the equilibrium effect of an intermittent, and sometimes costly, access to a centralized market. This literature started with Lagos and Rocheteau [2007] and Lagos and Rocheteau [2009]. These models allow for portfolio decisions, but the inter-dealer is assumed to be competitive and dealers do not keep any inventories.

Our model is also related to classical references on inventory risk such as Ho and Stoll [1980] and Ho and Stoll [1981]. These references do not consider, however, the feedback effect of the intermediation on the liquidity needs of the investors. This equilibrium effect is at the core of our analysis.

The explicit bargaining procedure that we devise means that our model is also related to Samuelson [1984], Grossman and Perry [1986], Mailath and Postlewaite [1990]. In these references, just like in the classical references on inventory risk, there is no feedback effect of the quoting strategy on the distribution of valuations.

References such as Blouin and Serrano [2001], Duffie and Manso [2007], Duffie et al. [2009], and , Duffie, Malamud, and Manso [2010] consider asymmetric information in decentralized markets. However, these references focus on common value asymmetry whereas we analyze a setting with private value asymmetric information. Also the references do not consider portfolio decisions.

To obtain the equilibrium expressions in closed-form, we assume that agents are only risk-averse with respect to certain risks. The same procedure was used, for instance, by Biais [1993], Duffie et al. [2007], Vayanos and Weill [2008], Gârleanu [2009]. Other references using "source-dependent" risk-aversions include Hugonnier et al. [2013] and Skiadas [2013].

Finally, in terms of formalism, the interaction between the distribution of types, the individual policies, and the value functions, means that our model is related to the literature on mean-field games, as introduced by Lasry and Lions [2007].

The outline of the paper is as follows. Section 3.2 describes the assets, investors, and other exogenous elements of our model. Section 3.3 solves for the optimal policy of an individual. Section 3.4 maps a certain trading pattern on the OTC market to a consistent cross-sectional distributional distribution of types. Section 3.5 solves for the equilibrium of the model and discusses its properties. Section 3.6 concludes.

3.2 Model

In this section, we present the various exogenous elements of our model economy.

Assets and Investors

In our model, investors trade bilaterally to share risks. Our model is based on Lo et al. [2004], from whom we borrow the specification of the trade motives, and on Duffie et al. [2005], from whom we borrow the meeting technology on the OTC market. The exact bargaining procedure that defines the trade details is original.

There are two assets. First, there is a risk-free bond freely traded and whose rate of return r is exogenously given. Second, there is a risky asset ("the stock") whose cumulated payouts

$$(D_t)_{t\geq 0}$$

is an arithmetic Brownian motion. Namely,

$$\mathrm{d}D_t = m_d\,\mathrm{d}t + \sigma_d \mathrm{d}B_t,\ t \geq 0,$$

with μ and σ being two constants and $(B_t)_{t\geq 0}$ being a Brownian motion.

The economy is populated by a normalized continuum of investors. The investors trade the stock for risk-sharing motives. Namely, each investor a receives an endowment

$$\left(\eta^a_t\right)_{t\geq 0}$$

whose dynamics are given by

$$\begin{cases} \mathrm{d}\eta^a_t & = & Z^a_t \mathrm{d}D_t \\ \mathrm{d}Z^a_t & = & \sigma_a \mathrm{d}B^a_t \end{cases},$$

with $\left(B^a_t\right)_{t\geq 0}$ being an "idiosyncratic" Brownian motion independent from the one driving the dividends of the stock. By *idiosyncratic* we mean that there is one such process per investor and that these processes are sufficiently independent for a version of the Strong Law of Large Numbers (SLLN) to hold cross-sectionally.

To sum up, the same aggregate risk factor drives the payouts of the stock and the endowment of the agents over the short-term. However, the level to which an endowment is exposed to this aggregate risk factor evolves in an idiosyncratic way.

Trading

Illiquidity on an OTC market materializes in that a counter-party is only infrequently available, does not necessarily want to trade the right quantity, and not necessarily at the right price. We capture the first of these aspects by assuming that an investor can only contact a counter-party at the jump times of a Poisson process with intensity λ. The counter-party who is contacted is drawn uniformly from across the population.

Once two investors are in contact, they must evaluate whether or not they wish to trade the stock and, if so, what the exact terms of the transaction should be.

On actual OTC markets, a common procedure to arrange a deal with a dealer is to ask the dealer for a quote and, assuming that the quote is deemed good enough, indicate how much of the asset one would like to either buy or sell. The quote usually consists of both a bid and an ask price.

For the trading in our model, we assume a stylized version of the previous procedure. Namely, once two investors are in contact, one of them quotes a binding price and the other one chooses a quantity to be exchanged at the quoted price. The exact bargaining procedure follows.

Bargaining 1:

Assume that the investors a (like "asks") just contacted the investor q (like "quotes"). These investors are identified with their "types" z_a, z_q, respectively. The distribution of the types across the population is μ.

(i) a asks q for a quote.

(ii) if *q finds it optimal to quote a price* **then**

1. q receives a signal s_a regarding the type of a. Namely,

$$s_a = Xz_a + (1 - X)\zeta, \tag{3.1}$$

   with $X \sim B(1,\tau)$ and $\zeta \sim \mu$ being independent of each other and of both z_a and z_q. Intuitively, the signal is exact with a probability τ which we call the *transparency* of the OTC market. In the other case, the signal is a pure noise.

2. q quotes a price $p = p(q, s_a)$ at which she is willing to trade with a

3. a chooses which quantity $q(a, p)$ she would like to buy (if $q \geq 0$) or sell (if $q < 0$).

4. q gives q units of the stock to a, a pays the amount $pq(a, p)$ to q

end

(iii) the two investors part ways.

Two comments are in order. First, when quoting the price p, the investor q may not be fully aware of the characteristics of a. This uncertainty regarding the valuation of one's counterparty is the type of opacity that our model captures.

Second, we assume that q quotes a unique price instead of both a bid and an ask price, as actual dealers would do. This assumption is made for the sake of tractability. This, said, as long as q has an accurate enough guess of a's valuation for the asset, q knows with some confidence whether the trade is going to be a buy or a sell. In particular, even if both a bid and an ask prices are quotes, only one of them is truly relevant.

Preferences

The investors maximizes their expected utility from consumption and have a utility function with constant absolute risk-aversion (CARA, or exponential, utility). Namely, an investor a solves the individual optimization problem

$$\sup_{(\tilde{c}_t)_{t\geq 0}} \mathrm{E}\left[\int_0^\infty e^{-\rho t} U(\tilde{c}_t)\, \mathrm{d}t\right] \tag{ip}$$

with the utility function

$$U(x) \triangleq -e^{-\gamma x}$$

over the "admissible" consumption policies. The constant $\gamma > 0$ is the coefficient of absolute risk-aversion. A consumption policy $(\tilde{c}_t)_{t\geq 0}$ is *admissible* if it satisfies two conditions.

1. The consumption policy can be financed, meaning that it satisfies the budget constraint

$$\mathrm{d}\tilde{w}_t^a = r\,\tilde{w}_t\mathrm{d}t - \tilde{c}_t\mathrm{d}t + \mathrm{d}\eta_t^a + \tilde{\theta}_t\mathrm{d}D_t - \tilde{P}_d\mathrm{d}\tilde{\theta}_t,\ t \geq 0. \tag{bc}$$

   In this last expression, the quantities with a tilde $(\tilde{\cdot})$ are endogenously chosen, whereas the other ones are fully exogenous. The interpretations of the endogenous quantities are as follows. $\tilde{w}_t^a$ denotes the amount invested in the bond and $\tilde{\theta}_t$ the number of stock shares held. The holdings $\tilde{\theta}_t$ can only be adjusted when another investor is met and, during such a meeting, both the change in holdings $\mathrm{d}\tilde{\theta}_t$ and the payment $\tilde{P}_d\,\mathrm{d}\tilde{\theta}_t$ are defined by the bargaining procedure described in Table 1. That is, $\tilde{P}_d$ is not a unique price in that it is contingent on the types of agents involved in a particular meeting.

2. The wealth process $(\tilde{w}_t)_{t\geq 0}$ satisfies the transversality condition

$$\lim_{T\to\infty} \mathrm{E}\left[e^{-r\gamma\tilde{w}_T}\right] = 0. \tag{tc}$$

   This regularity condition forbids the "financing" of consumption by an ever increasing amount of debt.

An agent is exposed to risky cash-flows both via her endowment and via her stock holdings. However, both of these exposures are driven by the same risk factor. For convenience and ease of interpretation, we thus define the *actual exposure* of the investor a as

$$z_t^a \triangleq Z_t^a + \theta_t$$

and rewrite the budget constraint as

$$\mathrm{d}\tilde{w}_t^a = r\,\tilde{w}_t\mathrm{d}t - \tilde{c}_t\mathrm{d}t + z_t^a\mathrm{d}D_t - P_d\mathrm{d}\theta_t.$$

The actual exposure follows a jump diffusion,

$$\mathrm{d}z_t^a = \mathrm{d}\theta_t + \sigma^a\,\mathrm{d}B_t^a, \tag{3.2}$$

with the jump part stemming from the trading and the diffusion part stemming form the idiosyncratic variation of the endowment exposure.

The actual exposure of an investor will define both her bargaining behavior and her value function. Consequently, the actual exposure of an investor is also referred to as the *type* of an investor. Also, during the bargaining procedure, the signal received by the quoter is about the type of her counter-party.

3.3 Individual Problem

A given investor takes as given the aggregate quantities of the model and chooses her consumption and bargaining policies to solve her individual problem (ip). Aggregating these individual "best responses" yields new aggregate quantities. When we will solve for an equilibrium, we will solve for a fixed point over these aggregate quantities.

In this section, we solve for the individual policies of the agents and do so by the dynamic programming approach, meaning by solving a Hamilton-Jacobi-Bellman (HJB) equation.

First, we define the *value function* at time t of an investor a by

$$V(t,w,z) \triangleq \sup_{(\tilde{c}_s)_{s\geq t}} \mathrm{E}\left[\int_t^\infty e^{-\rho(s-t)} U(\tilde{c}_s)\ \mathrm{d}s \,\middle|\, w_t^a = w, z_t^a = z\right],$$

with w_t^a standing for a's wealth at time t and z_t^a standing for a's type at time t. Let us take as given an optimal consumption policy $(c_t)_{t\geq 0}$ and make two assumptions. First, the environment is stationary, meaning that the beliefs regarding the aggregate quantities are constant over time. Second, in terms of expected utility, an agent is fully described by her current wealth w_t^a and current type z_t^a.[11] Then, one can write

$$\left(\int_0^t e^{-\rho s} U(c_s)\ \mathrm{d}s + e^{-\rho t} V(w_t, z_t)\right)_{t\geq 0} = \left(\mathrm{E}\left[\int_0^\infty e^{-\rho s} U(c_s)\ \mathrm{d}s \,\middle|\, \mathscr{F}_t^a\right]\right)_{t\geq 0}, \tag{3.3}$$

with $\mathscr{F}_t^a$ standing for the information available to a at time t. We left out the time as an argument of the value function because of the stationarity assumption.

As the process on the right-hand side of (3.3) is a martingale, so is the one on the left-hand side, and its expected rate of change must be zero. Now, assuming that the value function is regular enough for Itô's lemma for jump-diffusions to hold, the expected rate of change of the

[11] These assumptions will be justified ex-post.

process on the left-hand side is

$$\frac{1}{\mathrm{d}t}\mathrm{E}\left[\mathrm{d}\left(\int_0^t e^{-\rho s}U(c_s)\,\mathrm{d}s + e^{-\rho t}V(w_t,z_t)\right)\right]$$
$$= e^{-\rho t}\left(\begin{array}{l} U(c_s)-\rho V(w,z)+V_w(w,z)(rw-c+zm_d) \\ +\frac{1}{2}(V_{ww}(w,z)z^2\sigma^2+V_{zz}(w,z)\sigma_z^2) \\ +\lambda\,\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathbf{1}_{\{z_q\in A\}}\left(\begin{array}{c} V\left(w-qP\left(z_q,s_z\right),z+q\right) \\ -V(w,z)\end{array}\right)\right] \\ +\lambda\left[\begin{array}{c}\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[V\left(w+Q\left(z_a,p\right)p,z-Q\left(z_a,p\right)\right)\right] \\ -V(w,z)\end{array}\right]^+ \end{array}\right). \tag{3.4}$$

On the right-hand side, the first line corresponds to the utility from current consumption and the drift term of the value function. On this line, the consumption rate c_t is chosen by the investor.

The second line corresponds to the diffusion of the value function. There is no choice variable on this line.

The third line corresponds to the jump resulting from asking another investor for a quote. Namely, z_q is the type of the investor who was contacted and is drawn from the cross-sectional distribution of types μ. This distribution μ is taken as given by the individual investors. On this same line, s_z is the signal received by the potential quoter. In line with the definition (3.1),

$$s_z = X_z z + (1-X_z)\zeta_z,$$

with the two random variables $X_z \sim B(1,\tau)$ and $\zeta_z \sim \mu$ being independent of each other and of all the other random quantities. The set A appearing in the indicator function represents the types of the investors who are ready to offer a quote. This set is also taken as given. The function

$$\begin{array}{rcl} A\times\mathbb{R} & \to & \mathbb{R} \\ (z_q,s_z) & \mapsto & P\left(z_q,s_z\right) \end{array}$$

represents the quoting strategy adopted by the other investors, and is also taken as given. On this third line, the purchase q is chosen by the investor after observing the quote $P(z_q,s_z)$.

The fourth line corresponds to the jump resulting from receiving a quote request. Namely, z_a is the type of the investor who asked for a quote and is also distributed according to μ. The signal regarding z_a, s_a is given by

$$s_a = X_a z_a + (1-X_a)\zeta_a,$$

with $X_a \sim B(1,\tau)$, $\zeta_a \sim \mu$, and the same independence assumptions as above. The function

$$\begin{array}{rcl} \mathbb{R}^2 & \to & \mathbb{R} \\ (z_a, p) & \mapsto & Q\left(z_a, p\right) \end{array}$$

represents the purchase strategy adopted by the other investors, and is also taken as given. In this line, the quote p is chosen by the investor. The positive part in the expectation ("$[\cdot]^+$") represents the optimal decision of quoting or not.

Combining the martingale property of the processes in (3.3), the expected dynamics in (3.4), and the intuition that the optimal policy should be locally given by a maximization of these expected dynamics over the choice variables, we derive the HJB equation for the individual problem (ip). Namely,

$$\begin{aligned} \rho V(w,z) = & \sup_{\tilde{c}} \{U(\tilde{c}_s) - V_w(w,z)\tilde{c}\} \\ & + V_w(w,z)(rw + zm_d) \\ & + \frac{1}{2}\left(V_{ww}(w,z)z^2\sigma^2 + V_{zz}(w,z)\sigma_z^2\right) \\ & + \lambda \mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathbf{1}_{\{z_q\in A\}}\begin{pmatrix} \sup_{\tilde{q}} V\left(w - \tilde{q}P\left(z_q,s_z\right), z+\tilde{q}\right) \\ -V(w,z) \end{pmatrix}\right] \\ & + \lambda \begin{bmatrix} \mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\sup_{\tilde{p}} \mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\left. V\left(w + Q\left(z_a,\tilde{p}\right)\tilde{p}, z - Q\left(z_a,\tilde{p}\right)\right)\right| s_a\right]\right] \\ -V(w,z) \end{bmatrix}^+, \end{aligned} \tag{3.5}$$

with the random variables z_q, s_z, z_a, s_a, and set A satisfying the same distributional assumptions as above.

To analyze the HJB equation (3.5), we proceed in two steps. First, we assume a certain functional form ("Ansatz") for the solution to (3.5). Then, for tractability reasons, we focus on a certain asymptotic case.

Assumption 23. The value function can be written as

$$V(w,z) = -\exp\{-\alpha(w + v(z) + \bar{v})\},$$

with $\alpha > 0$, $v \in \mathscr{C}^2$ and $\bar{v} \in \mathbb{R}$.

Such a functional form is common in models of consumption-portfolio choice with CARA investors. See, among many others, Wang [1994], Duffie et al. [2007], or Gârleanu [2009]. Further, in an asymptotic case, this assumption will be justified ex-post by an explicit solution.

Note that the function $v(\cdot)$ and the constant $\bar{v}$ cannot be identified independently that $\bar{v}$ is only introduced for convenience.

With Assumption 23, the various derivatives appearing in (3.5) are all proportional to the value function itself. Namely,

$$\begin{aligned} V_w(w,z) &= (-\alpha)V(w,z) \\ V_{ww}(w,z) &= \alpha^2 V(w,z) \\ V_z(w,z) &= \left(-\alpha v'(z)\right)V(w,z) \\ V_{zz}(w,z) &= \left(\left(-\alpha v'(z)\right)^2 - \alpha v''(z)\right)V(w,z). \end{aligned} \tag{3.6}$$

This homogeneity of the problem will simplify its treatment. First, we can characterize the optimal consumption.

Lemma 24. *The optimal consumption in the HJB equation* (3.5) *is*

$$c = \frac{\alpha}{\gamma}(w + v(z) + \bar{v}) - \frac{1}{\gamma}\log\left(\frac{\alpha}{\gamma}\right).$$

It corresponds to a utility

$$U(c) = \frac{\alpha}{\gamma}V(w,z),$$

which is thus proportional to the value function.

Proof. The proofs are in Appendix D. □

Injecting the expressions for the derivatives of the value function in (3.6) and the optimal consumption stated in Lemma 24 into the HJB equation (3.5), and simplifying by $V(w,z)$, which is negative, yields

$$\begin{aligned} \rho =& \frac{\alpha}{\gamma} + \alpha\left(\frac{\alpha}{\gamma}(w + v(z) + \bar{v}) - \frac{1}{\gamma}\log\left(\frac{\alpha}{\gamma}\right)\right) \\ &- \alpha(rw + zm_d) \\ &+ \frac{1}{2}\left(\alpha^2 z^2\sigma^2 + \left(\left(-\alpha v'(z)\right)^2 - \alpha v''(z)\right)\sigma_z^2\right) \\ &+ \lambda \mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathbf{1}_{\{z_q\in A\}}\left(\inf_{\tilde{q}}\frac{V\left(w - \tilde{q}P_d\left(z_q,s_z\right), z+\tilde{q}\right)}{V(w,z)} - 1\right)\right] \\ &+ \lambda\left[\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\inf_{\tilde{p}}\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\left.\frac{V\left(w + Q\left(z_a,\tilde{p}\right)\tilde{p}, z - Q\left(z_a,\tilde{p}\right)\right)}{V(w,z)}\right| s_a\right]\right] - 1\right]^+. \end{aligned} \tag{3.7}$$

As this equation must hold for any wealth w and as, by Assumption 23, $\alpha > 0$,

$$\alpha\left(\frac{\alpha}{\gamma} - r\right)w = 0$$

$$\Leftrightarrow \qquad \alpha = r\gamma. \tag{3.8}$$

Now, injecting (3.8) into (3.7), choosing

$$\bar{v} \triangleq \frac{1}{r\gamma}\left(\frac{\rho}{r} - 1 + \log(r)\right)$$

to get rid of the constant terms, and normalizing by $-\alpha = -r\gamma$, yields

$$\begin{aligned} r\,v(z) =& z m_d - \frac{1}{2}\left(r\gamma z^2\sigma^2 + \left(r\gamma\left(v'(z)\right)^2 - v''(z)\right)\sigma_z^2\right) \\ &+ \frac{\lambda}{-r\gamma}\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathbf{1}_{\{z_q\in A\}}\left(\inf_{\tilde{q}} e^{-r\gamma(-\tilde{q}P_d(z_q,s_z)+v(z+\tilde{q})-v(z))} - 1\right)\right] \\ &+ \frac{\lambda}{-r\gamma}\left[\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\inf_{\tilde{p}}\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[e^{-r\gamma(Q(z_a,\tilde{p})\tilde{p}+v(z-Q(z_a,\tilde{p}))-v(z))}\middle| s_a\right]\right] - 1\right]^+. \end{aligned}$$

Rearranging the terms representing the trading now yields

$$\begin{aligned} r\,v(z) =& z m_d - \frac{1}{2}\left(r\gamma z^2\sigma^2 + \left(r\gamma\left(v'(z)\right)^2 - v''(z)\right)\sigma_z^2\right) \\ &+ \lambda\,\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathbf{1}_{\{z_q\in A\}}\sup_{\tilde{q}}\frac{1 - e^{-r\gamma(-\tilde{q}P_d(z_q,s_z)+v(z+\tilde{q})-v(z))}}{r\gamma}\right] \\ &+ \lambda\left[\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\sup_{\tilde{p}}\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\frac{1 - e^{-r\gamma(Q(z_a,\tilde{p})\tilde{p}+v(z-Q(z_a,\tilde{p}))-v(z))}}{r\gamma}\middle| s_a\right]\right]\right]^+. \end{aligned} \tag{3.9}$$

In this last equation (3.9), the first line of the right-hand side balances the instantaneous benefits (expected payouts) and costs (variance of payouts scaled by the risk-aversion) of having a certain exposure z to the aggregate risk factor. The second and third line corresponds to the benefits induced by the possibility to adjust one's exposure by trading the stock.

Analyzing equation (3.9) is difficult because of the non-linearity of the terms related to trading, those on the second and third line. This non-linearity itself stems from the risk-aversion toward the non-fundamental risks.

For the sake of tractability, and following arguments in Duffie et al. [2007], Vayanos and Weill [2008], and, to mention two particularly transparent examples, Biais [1993] and Gârleanu [2009], we will make the investors risk-neutral with respect to the trading risks while maintaining the risk-aversion with respect to the fundamental risk.

The proper way to achieve this "focused risk-aversion" is to let the risk-aversion coefficient go to zero,

$$\gamma \to 0,$$

while scaling up the fundamental aggregate risk

$$\sigma \sim \frac{1}{\sqrt{\gamma}} \xrightarrow{\gamma \to 0} +\infty.$$

As a result, the "quantity" of fundamental risk contained in any stock holding is maintained, but any other type of risk-aversion vanishes. The next assumption formalizes the asymptotic case that we will characterize explicitly.

Assumption 25. The volatility of the dividends is inversely proportional to the risk-aversion of the investors. Namely,

$$\sigma \stackrel{\Delta}{=} \frac{1}{\sqrt{\gamma}}\bar{\sigma},$$

with $\bar{\sigma} > 0$. Further, we assume that

$$v(z) = v_0(z) + \mathcal{O}(\gamma),$$

implicitly assuming that a solution $v(\cdot)$ to (3.9) exists for any sufficiently small value of γ.

As it turns out, focusing on the asymptotic behavior of the model significantly simplifies the analysis. Combining the HJB equation (3.9) and Assumption 27 characterizes the asymptotic function $v_0(\cdot)$ as a solution to

$$\begin{aligned} r v_0(z) =& z m_d - \frac{1}{2}\left(r\gamma z^2\sigma^2 - v_0''(z)\sigma_z^2\right) \\ &+ \lambda \mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathbf{1}_{\{z_q\in A\}} \sup_{\tilde{q}} \left(-\tilde{q}P_d\left(z_q,s_z\right) + v_0\left(z+\tilde{q}\right) - v_0(z)\right)\right] \\ &+ \lambda\left[\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\sup_{\tilde{p}} \mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\, Q\left(z_a,\tilde{p}\right)\tilde{p} + v_0\left(z - Q\left(z_a,\tilde{p}\right)\right) - v_0(z) \middle| \, s_a\right]\right]\right]^+ . \end{aligned} \tag{3.10}$$

We can characterize $v_0(\cdot)$ explicitly and, following the examples of Biais [1993], Duffie et al. [2007], Vayanos and Weill [2008], and Gârleanu [2009], we will focus on the analysis of the asymptotic, but much more tractable, value function $v_0(\cdot)$ instead of the general $v(\cdot)$. For convenience, we will thus abuse our notations and, from now on, write $v(\cdot)$ for $v_0(\cdot)$.

At this stage, we would like to state more formally what we are searching for.

The equation (3.10) relies on beliefs regarding two types of aggregate quantities. These aggregate quantities are, first, the cross-sectional distribution of types μ and, second, the quoting and purchasing policies $P(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ adopted by the other investors. The combination of these beliefs and the equation (3.5) define new, individually optimal, policies. Then, the aggregation of these individually optimal policy defines a certain type dynamics. Finally, a certain stationary distribution of types results from the type dynamics.

We want to solve for a rational expectations equilibrium of the model, meaning that we want the beliefs and the actual quantities to be consistent. We solve for such a rational expectations equilibrium in two steps. First, we take the type distribution as exogenous and ensure the rationality of the beliefs regarding the quoting and purchasing policies. We call such a solution, conditional on the type distribution, a “partial equilibrium” of the model. Then, we will ensure the rationality of the beliefs regarding the type distribution, and this will define an “equilibrium” of the model.

The formal definition of a partial equilibrium follows.

Definition 26 (Partial Equilibrium)**.** Let a cross-sectional distribution of types μ be given. Then, a partial equilibrium of the model consists of a triplet of functions and a set $A \subset \mathbb{R}^2$. The three functions are

$$z \mapsto v(z)$$

that describes how the value function of an investor depends on her type,

$$P : (z, s) \mapsto P(z, s),$$

that describes the quote provided by an investor of type z after receiving the signal s, and

$$Q : (z, p) \mapsto V(w, z)$$

that describes the number of shares purchased by an investor of type z after receiving a quote p.

The partial equilibrium quantities must be a “best-response” to themselves. Namely,

1. $v(\cdot)$ satisfies the HJB equation (3.9), given A, $Q(\cdot,\cdot)$, and $P(\cdot,\cdot)$;
2. the purchasing policy $Q(\cdot,\cdot)$ satisfies

   $$Q(z, p) \in \arg\max_{\tilde{q}} \left(-\tilde{q} P_d\left(z_q, s_z\right) + v\left(z + \tilde{q}\right) - v(z)\right),$$

   meaning that it is optimal, given $v(\cdot)$;
3. the set A satisfies

   $$A = \left\{ z : \mathrm{E}^{\mathscr{L}(z_a, s_a)} \left[\sup_{\tilde{p}} \mathrm{E}^{\mathscr{L}(z_a, s_a)} \left[Q\left(z_a, \tilde{p}\right) \tilde{p} + v\left(z - Q\left(z_a, \tilde{p}\right)\right) \middle| \, s_a \right] \right] \geq v(z) \right\},$$

meaning that it contains the types of the investor who are willing to issue a quote, given $Q(\cdot,\cdot)$ and $v(\cdot)$;

4. the quoting policy $P(\cdot,\cdot)$ satisfies

$$P(z, s_a) \in \arg\max_{\tilde{p}} \mathrm{E}^{\mathscr{L}(z_a, s_a)} \left[Q\left(z_a, \tilde{p}\right) \tilde{p} + v\left(z - Q\left(z_a, \tilde{p}\right)\right) \middle| s_a \right]$$

on the set $A \times \mathbb{R}$, meaning that it is optimal, given A, $Q(\cdot,\cdot)$ and $v(\cdot)$.

The equilibrium is partial because there is no connection between, on the one hand, the trading pattern induces by $P(\cdot,\cdot)$, $Q(\cdot,\cdot)$, and A and, on the other hand, the distribution of types μ.

Note that, strictly speaking, we are interested in the solutions to the individual problem (ip) and not in the solutions to the HJB equation (3.9). As these two sets need not be identical, this calls for a verification argument.

Proposition 27. *Let us assume that the transparency of the market is high enough, meaning that*

$$\tau \in \left[\frac{\sqrt{3}}{2}, 1\right] \approx [0.866, 1]\,, \tag{3.11}$$

and let us take as given the cross-sectional distribution of types μ *and write* $\mathscr{M}$ *for its mean and* $\mathscr{V}$ *for its variance. Then, there exists a partial equilibrium for which the value functions are characterized by the quadratic function*

$$v(z) = v_0 + v_1 z + v_2 z^2,$$

with

$$\begin{cases} v_0 &= \frac{\gamma\sigma^2\left(4\lambda\mathscr{M}^2\left((\tau-3)\tau^2+3\right)-9\sigma_z^2-4\lambda\left((\tau-4)\tau^2+2\right)\mathscr{V}\right)}{8\lambda\left((\tau-3)\tau^2+3\right)+18r} \\ v_1 &= \frac{\mu}{r} - \frac{4\gamma\lambda\mathscr{M}\sigma^2\left((\tau-3)\tau^2+3\right)}{4\lambda\left((\tau-3)\tau^2+3\right)+9r} \\ v_2 &= -\frac{9\gamma r\sigma^2}{8\lambda\left((\tau-3)\tau^2+3\right)+18r} \end{cases}.$$

The corresponding optimal purchasing policy is

$$Q : (z, p) \mapsto \frac{p - v_1}{2v_2} - z.$$

All the agents quote a price when being asked for one, meaning that

$$A \overset{(\Delta)}{=} \left\{ z : \mathrm{E}^{\mathscr{L}(z_a, s_a)} \left[\sup_{\tilde{p}} \mathrm{E}^{\mathscr{L}(z_a, s_a)} \left[Q\left(z_a, \tilde{p}\right) \tilde{p} + v\left(z - Q\left(z_a, \tilde{p}\right)\right) \middle| s_a \right] \right] \geq v(z) \right\}$$
$$= \mathbb{R}.$$

And the function

$$P : (z, s_a) \mapsto v_1 + 2v_2 \left(\frac{1}{3} z + \frac{2}{3} (\tau s_a + (1-\tau)\mathscr{M}) \right)$$

describes the optimal quoting policy.

Remark 28. Without the assumption (3.11), no quadratic partial equilibrium exists. In the proof of Proposition 27, this assumption is critical in step (iv). If this assumption is relaxed, the term describing the utility benefits resulting from being asked for a quote will not be quadratic in the current type anymore, and the quadratic assumption in step (i) will not be consistent.

In order to characterize an equilibrium of the model, we still need the cross-sectional distribution of types or, as seen in Proposition 27, its two first moments. The next corollary is the first step in this direction.

Corollary 29. *The dynamics of the type z of a given agent a is*

$$\begin{aligned} \mathrm{d}z_t &= \sigma_z \, \mathrm{d}B_t \\ &+ \left(\begin{array}{rl} X_{r,t} & \left(\frac{1}{3} z_{q,t} + \frac{2}{3} (\tau z_{t-} + (1-\tau)\mathscr{M})\right) \\ + \left(1 - X_{r,t}\right) & \left(\frac{1}{3} z_{q,t} + \frac{2}{3} \left(\tau \zeta_{r,t} + (1-\tau)\mathscr{M}\right)\right) \\ - & z_{t-} \end{array} \right) \mathrm{d}N_t^r \\ &+ \left(\begin{array}{rl} X_{q,t} & \left(z_{r,t} + \frac{2}{3} \left(\tau z_{t-} - \left(\tau z_{r,t} + (1-\tau)\mathscr{M}\right)\right)\right) \\ + \left(1 - X_{q,t}\right) & \left(z_{r,t} + \frac{2}{3} \left(\tau z_{t-} - \left(\tau \zeta_{q,t} + (1-\tau)\mathscr{M}\right)\right)\right) \\ - & z_{t-} \end{array} \right) \mathrm{d}N_t^q, \end{aligned} \tag{3.12}$$

with the following distributional assumptions.

1. *N^r is a Poisson process with jump intensity λ. N^c jumps when a requests a quote from another investor.*
2. *$z_{q,t} \sim \mu$ is the type of the investor from whom a requests a quote at time t.*
3. *$X_{r,t} \sim B(1, \tau)$ is the Bernoulli random variable indicating whether the signal about a at time t is correct ($X_{a,t} = 1$) or uninformative ($X_{a,t} = 0$).*
4. *$\zeta_{r,t} \sim \mu$ is the uninformative signal about a.*

5. *N^q is a Poisson process with jump intensity λ. N^q jumps when a is asked for a quote by another investor.*

6. *$z_{r,t} \sim \mu$ is the type of the investor who requested a quote from a at t.*

7. *$X_{r,t} \sim B(1,\tau)$ is the Bernoulli random variable indicating whether a observes the current type $z_{r,t}$ or the agent who requested a quote at time t ($X_{r,t} = 1$) or an uninformative signal ($X_{r,t} = 0$).*

8. *$\zeta_r \sim \mu$ is the uninformative signal about the agent who requested a quote.*

Furthermore, the processes

$$B, N^r, N^q$$

and random variables

$$\left(z_{q,t}, X_{r,t}, \zeta_{r,t}, z_{r,t}, X_{q,t}, \zeta_{q,t}\right)_{t\geq 0}$$

are all independent of each others.

3.4 Stationary Type Distribution

The calculation of the individual value functions in Section 3.3 relies on exogenous beliefs regarding the distribution of types across the investors. However, these individual value functions themselves induce a certain trading pattern on the OTC market, and this trading pattern generates a certain type distribution. In this section we intend to make the beliefs regarding the type distribution rational. We formalize the rationality of the beliefs with the following definition.

Definition 30 (Consistent Type Distribution)**.** A distribution of types μ is consistent if

1. the trading pattern induced by the partial equilibrium corresponding to μ generates a stationary distribution of types $\mu^{\text{out}}(\mu)$;

2. the assumed and actual distributions of types are identical, meaning that $\mu = \mu^{\text{out}}(\mu)$.

As it turns out, there is a unique consistent distribution of types.[12]

[12] The exact sense in which the trading pattern "generates" a stationary type distribution is clear from the proof of the next proposition.

Proposition 31. *There exists a unique equilibrium stationary distribution of types* μ. μ *solves the equation*

$$\hat{\mu}(w) = \frac{1}{1 + \frac{\frac{1}{2}\sigma_z^2}{2\lambda} w^2} \left(\begin{array}{rccc} \frac{\tau}{2} & e^{i\frac{2}{3}(1-\tau)\mathscr{M}w} & \hat{\pi}\left(\frac{1}{3}w\right) & \hat{\pi}\left(\frac{2}{3}tw\right) \\ +\frac{1-\tau}{2} & e^{i\frac{2}{3}(1-\tau)\mathscr{M}w} & \hat{\pi}\left(\frac{1}{3}w\right) & \hat{\pi}\left(\frac{2}{3}tw\right) \\ \frac{\tau}{2} & e^{-i\frac{2}{3}(1-\tau)\mathscr{M}w} & \hat{\pi}\left(\frac{2}{3}w\right) & \hat{\pi}\left(\left(1-\frac{2}{3}t\right)w\right) \\ +\frac{1-\tau}{2} & e^{-i\frac{2}{3}(1-\tau)\mathscr{M}w} & \hat{\pi}\left(\frac{2}{3}w\right) & \hat{\pi}(w)\,\hat{\pi}\left(-\frac{2}{3}tw\right) \end{array} \right), \quad w \in \mathbb{R}, \tag{3.13}$$

with $\hat{\mu}(w)$ *being the Fourier transform of* μ. *Namely, if* $z \sim \mu$, *then*

$$\hat{\mu}(w) \triangleq \mathrm{E}\left[e^{iwz}\right].$$

Furthermore, the first two moments of μ *are*

$$\begin{aligned} \mathrm{E}[z] &= S \\ \mathrm{Var}[z] &= \frac{9\sigma_z^2}{4\lambda\left(1+\tau^2\right)}. \end{aligned}$$

Proof. See Proof 62 in the appendix. □

At this stage, we can already describe certain characteristics of the distribution of trades.

Corollary 32. *The average transaction price is*

$$\frac{\mu}{r} - \gamma S \sigma^2.$$

The variance of the transaction prices is

$$\frac{81\gamma^2 r^2 \sigma^4 \sigma_z^2 (4\tau+1)}{\lambda\left(\tau^2+1\right)\left(8\lambda\left((\tau-3)\tau^2+3\right)+18r\right)^2}$$

and is increasing in the transparency levels whenever

$$\lambda \geq \frac{6\left(131\sqrt{3}-101\right)}{20641} r \approx 0.037r$$

and the transparency τ *is high enough. The variance of the transaction sizes is*

$$\frac{\left(10-8\tau^2\right)\sigma_z^2}{4\lambda\left(1+\tau^2\right)}$$

and is decreasing over the relevant range of transparency levels.

Finally, an ordinary least-squares regression of the transaction prices against the transaction sizes yields the constant coefficient

$$\frac{\mu}{r} - \gamma S \sigma^2$$

and the slope coefficient

$$\frac{9}{10}\frac{2}{3}\left(\frac{1}{3\nu_2}(4\tau+1) - 2\nu_2\tau^2\right).$$

The slope coefficient is increasing in τ over the relevant range when λ is not too small when compared to r.[13]

Interestingly, the variance of the transaction prices is typically increasing in the transparency of the market. It is true that a more transparent market leads to a more efficient allocation of the stock, to smaller liquidity needs, and to more homogeneous valuations across the population of investors. This should tend to decrease the cross-sectional variance of the transaction prices. However, more transparency leads to predatory quotes, which increases the inventory costs. This second effect increases the unit price of a transaction of a given size, increases the dispersion of prices, and dominates the first effect.

3.5 Equilibrium

Putting together the results of Section 3.3 regarding the individual optimality and Section 3.4 regarding the type distribution, the characterization of an "equilibrium" of the model is immediate. We first formally define our equilibrium concept.

Definition 33 (Equilibrium)**.** An equilibrium of the model consists of a consistent distribution of types μ, in the sense of Definition 30, and the corresponding partial equilibrium, in the sense of Definition 26.

Corollary 34. *There exists exactly one quadratic equilibrium.*

Proof. Immediate from Proposition 27 and Proposition 30. □

[13] The exact condition is

$$2\left(71+40\sqrt{3}\right)\lambda^3 + 108r^3\left(3\sqrt{3}\gamma^2\sigma^4 - 8\right) + 9\lambda r^2\left(\left(48\sqrt{3}-9\right)\gamma^2\sigma^4 - 64\sqrt{3}+72\right) + 120\left(3+\sqrt{3}\right)\lambda^2 r > 0$$

and is satisfied when we let either $\lambda \to \infty$ or $r \to 0$.

We can characterize the links between the expected utility of the investors and the transparency of the OTC market.

Corollary 35. *The relationship between transparency, as measured by the parameter τ measuring the quality of the signal, and expected utility, as measured by the function $v(\cdot)$, is ambiguous. Namely,*

$$\lim_{\tau\to 1}\frac{\partial}{\partial\tau}v(z)\begin{cases} >0 & , \quad |z-\mathscr{M}| \quad < \quad \frac{\sqrt{r\lambda(9r+\lambda)\sigma_z^2}}{\sqrt{6}r\lambda} \\ =0 & , \quad |z-\mathscr{M}| \quad = \quad \frac{\sqrt{r\lambda(9r+\lambda)\sigma_z^2}}{\sqrt{6}r\lambda} \\ <0 & , \quad |z-\mathscr{M}| \quad > \quad \frac{\sqrt{r\lambda(9r+\lambda)\sigma_z^2}}{\sqrt{6}r\lambda} \end{cases}.$$

In particular, more transparency benefits those investors with a moderate exposure to the aggregate risk but more opacity benefits those investors having either a sufficiently large or sufficiently low exposure. Overall, more transparency is socially desirable in the sense that

$$\partial_\tau \,\mathrm{E}^{\mu(z)}\left[v(z)\right] = \frac{9\gamma\sigma^2\sigma_z^2\tau}{4\lambda\left(1+\tau^2\right)^2} > 0.$$

The intuition behind the last proposition is that transparency increases the inventory costs.[14] This happens via two different channels. On the one hand, transparency exposes any given investor with large inventories to predatory pricing. On the other hand, as transparency improves the allocative efficiency of the market, it becomes more difficult for an investor with an excessive exposure to find a counterparty with large and opposite liquidity needs. This second effect exacerbates the first one and is driven by the endogenous distribution of types.

These results imply that the effect of transparency on the value function of the investors is ambiguous. First, on average, investors benefit from an increase of the transparency and the resulting improvement in the allocative efficiency. In particular, transparency is welfare improving. However, those investors with a sufficiently large (positive) or small (negative) exposure find it increasingly costly to liquidate this exposure and would benefit from a more opaque market. We characterize explicitly the exposure levels starting from which investors prefer opacity to transparency. This result is in line with the heterogeneous reactions to the introduction of TRACE, with quite negative reactions of many institutional investors. Indeed, as our model predicts, transparency is a disadvantage for agents facing large inventory risk, such as, e.g., certain institutional investors.

[14] We define the inventory costs as the reservation value of an investor who deviates from a zero risk exposure.

3.6 Conclusion

We study a general equilibrium model in which agents share risks by trading on an OTC market. Both the transaction size and the transaction price are bargained bilaterally, and we analyze the equilibrium effect of asymmetric information regarding one's counter-party liquidity needs. We call the quality of this information the transparency of the market. We solve for both the value functions and the moments of the endogenous distribution of types in closed-form. Increased transparency has two main effects on the equilibrium of the model. On the one hand, it makes the asset allocation more efficient. On the other hand, it induces agents to adopt predatory quoting policies. These two effects both tend to increase the inventory costs, and we show how transparency is beneficial to those agents with moderate liquidity needs but detrimental to the rest of the population. We characterize the threshold starting from which investors value opacity in closed-form. Overall, however, more transparency is welfare improving. Our conclusions are in line with a number of sources documenting the mixed effects of transparency on the liquidity of certain OTC markets.[15]

One natural extension of our model is to consider two classes of agents. Agents in the first class, representing the dealers, trade both among themselves and with the agents of the second class. Agents in the second class, representing the end-users, can only trade with the dealers. In this setting, one can analyze the heterogeneous effects of transparency. Indeed, post-trade transparency makes the valuations of the asset across the end-users more homogeneous, but, for the dealers and as in the current model, it makes the information regarding one's counter-party inventory more accurate. Transparency may impact the entry decision of dealers. Understanding the interaction between these two sides of transparency is critical if one wants to evaluate the new regulatory reforms regarding the transparency of OTC markets.

[15] See Appendix C.

3.7 Figures

notation	parameter	value
r	risk-free rate	0.037
μ	expected dividend of the risky asset	0
σ	volatility of the dividends of the risky asset	0.285
σ_z	volatility of the idiosyncratic exposure	1
γ	absolute coefficient of risk-aversion	5
M_1	net supply of the risky asset	0
λ	meeting intensity on the OTC market	50

Table 3.1: Baseline parameter values for the numerical examples.

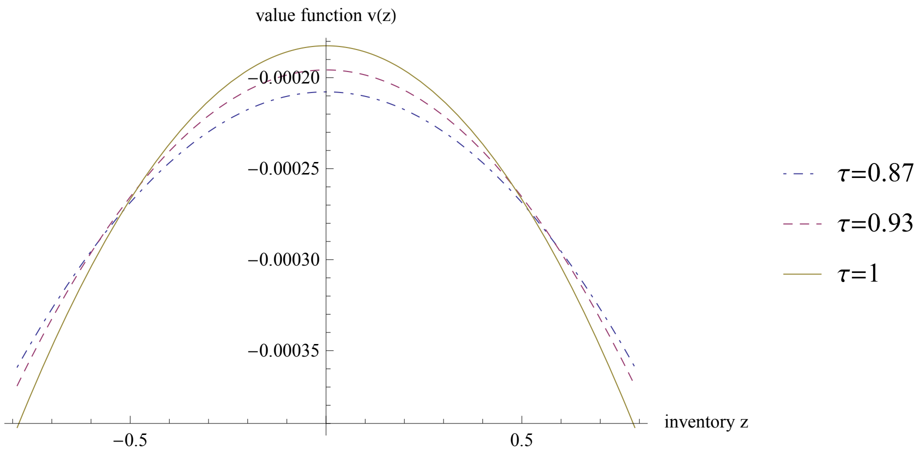

Figure 3.1: Value function as a function of the inventories for three levels of transparency τ. Note the non-monotone effect of transparency on the value function. The baseline parameters value are in Table 3.1.

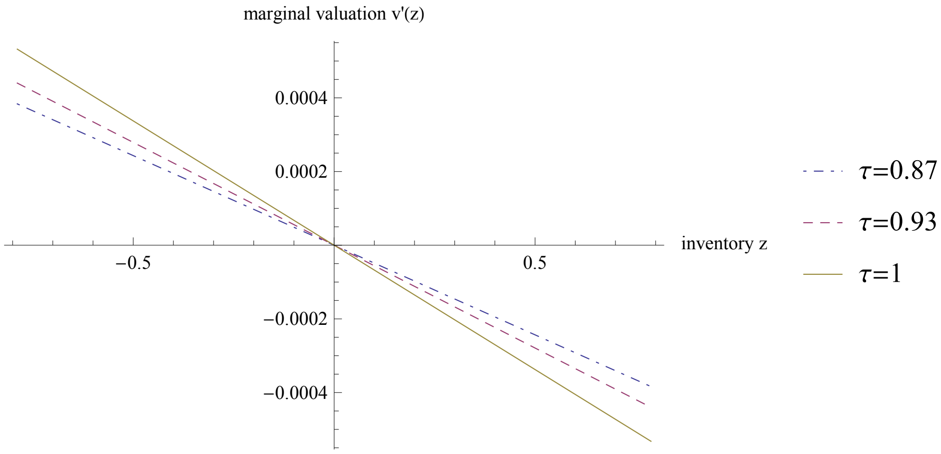

Figure 3.2: Marginal valuation of the risky asset as a function of the inventories for three levels of transparency τ. Transparency increases the dispersion of the valuations. The baseline parameters value are in Table 3.1.

A Proofs for Chapter 1

Asymptotic Behavior of the Equilibrium

I cannot solve for the unique stationary equilibrium of the model in closed form. The technical difficulty preventing it are the exponential terms related to the jump risks in the HJB equation (1.20). Closed-form expressions can, however, be obtained in the asymptotic case analyzed in Section 1.6.[1]

I will also assume a relatively liquid OTC market. I would like to justify this assumption. An investor will only bother to enter an illiquid market if she expects to amortize the costly process of building up, and liquidating, a position over a reasonably long holding period. This intuition is formalized in Vayanos and Wang [2007] within a search model of asset pricing, and goes back to Amihud and Mendelson [1986] for a setting with exogenous transaction costs.

This suggests that I may assume the illiquidity level $\xi = 1/\Lambda$ to be small relatively to $1/\lambda_{12}$ and $1/\lambda_{21}$, which are the average times (continuously) spent with a high or a low valuation.

Proposition 36. *I assume both*

$$\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right) > 0, \tag{A.1}$$

which is the condition (1.33) *in Proposition 10, and*

$$\mu_2 > \frac{S_d}{\Theta},$$

[1] The asymptotic analysis in Section 1.6 amounts to letting the investors become nearly risk-neutral with respect to the jump risks but maintain their risk-aversion toward the diffusion risks. The exact definition of the asymptotic case is in Equation (1.47) and (1.48).

which is condition (1.31) *in Section 7. Then, the price P_d bargained on the OTC market satisfies*

$$P_d = P_{d,W} - \left\{\frac{1}{\Lambda}\right\}\left\{\frac{\eta_0\left(r+2\delta_\mu\right)+\lambda_{21}}{2\eta_\Theta\left(\mu_2-\frac{S_d}{\Theta}\right)}\right\}\left\{\gamma\frac{\det\left(\left(\begin{array}{c:c} e_d & e_c\end{array}\right)\right)\cdot\det\left(\left(\begin{array}{c:c} e_1-e_2 & e_c\end{array}\right)\right)}{\Sigma_{cc}}\right\} + o\left(\frac{1}{\Lambda}\right)+\mathcal{O}(\gamma), \tag{A.2}$$

with the Walrasian price

$$P_{d,W} = \frac{\kappa\,(2\Theta)-\kappa\,(20)}{\Theta r}.$$

Also, the sensitivity

$$\delta_\mu \overset{(\Delta)}{=} \lim_{\frac{1}{\Lambda}\to 0}\frac{\partial\left(\mu(1h)\right)}{\partial\left(\frac{1}{\Lambda}\right)}$$

of the type distribution to the search friction was defined in Proposition 8.[2]

Proof 37 (Proof of Proposition 36)**.** Under the assumption (A.1) of the statement, and recalling Proposition 1.5, the only profitable type of trade on the OTC market is a sale by a 1Θ-investor to a 20-investor. As a result, the HJB equations (1.20) become

$$\left\{\begin{array}{lcllll} ra(10) & = & \kappa(10) & +\lambda_{12}\left(\frac{e^{-r\gamma(a(20)-a(10))}-1}{-r\gamma}\right) & & \\ ra(1\Theta) & = & \kappa(1\Theta) & +\lambda_{12}\left(\frac{e^{-r\gamma(a(2\Theta)-a(1\Theta))}-1}{-r\gamma}\right) & +2\Lambda\mu(20)\frac{\chi(\eta_\Theta,\epsilon_{1\Theta}(a))}{-r\gamma} \\ ra(20) & = & \kappa(20) & +\lambda_{21}\left(\frac{e^{-r\gamma(a(10)-a(20))}-1}{-r\gamma}\right) & +2\Lambda\mu(1\Theta)\frac{\chi(\eta_0,\epsilon_{20}(a))}{-r\gamma} \\ ra(2\Theta) & = & \kappa(2\Theta) & +\lambda_{21}\left(\frac{e^{-r\gamma(a(1\Theta)-a(2\Theta))}-1}{-r\gamma}\right) & \end{array}\right. \tag{A.3}$$

As

$$\chi\left(\eta_0,x\right) = -r\gamma\eta_0 x + o(\gamma),$$

this last system of equations becomes

$$\left\{\begin{array}{lcllll} ra(10) & = & \kappa(10) & +\lambda_{12}\left(a(20)-a(10)\right) & +\mathcal{O}(\gamma) \\ ra(1\Theta) & = & \kappa(1\Theta) & +\lambda_{12}\left(a(2\Theta)-a(1\Theta)\right) & +2\Lambda\mu(20)\eta_\Theta\epsilon_{1\Theta}\left(a\right)+\mathcal{O}(\gamma) \\ ra(20) & = & \kappa(20) & +\lambda_{21}\left(a(10)-a(20)\right) & +2\Lambda\mu(1\Theta)\eta_0\epsilon_{20}\left(a\right)+\mathcal{O}(\gamma) \\ ra(2\Theta) & = & \kappa(2\Theta) & +\lambda_{21}\left(a(1\Theta)-a(2\Theta)\right) & +\mathcal{O}(\gamma) \end{array}\right. \tag{A.4}$$

in the asymptotic case described by Equations (1.47) and (1.48). In this same asymptotic case,

[2] Recall that these results only hold under the assumption 1.31 regarding the marginal buyer.

Equation (1.17) significantly simplifies as well and the bargained price P_d is

$$
\begin{aligned}
P_d &= \eta_\Theta \left(a(2\Theta) - a(20)\right) + \eta_0 \left(a(1\Theta) - a(10)\right) + o(\gamma) \\
&= \left(a(2\Theta) - a(20)\right) - \eta_0 \begin{pmatrix} (a(2\Theta) - a(20)) \\ \quad - (a(1\Theta) - a(10)) \end{pmatrix} + \mathcal{O}(\gamma).
\end{aligned}
\tag{A.5}
$$

Using the expressions in (A.4) to reformulate (A.5) yields

$$
P_d = \frac{1}{r}\left(\kappa(2\Theta) - \kappa(20)\right) - \frac{\lambda_{21} + 2\Lambda\mu(1\Theta)\eta_0 + \eta_0 r}{r + 2\Lambda\left(\eta_\Theta\mu(20) + \eta_0\mu(1\Theta)\right)} + \mathcal{O}(\gamma). \tag{A.6}
$$

Finally, combining the asymptotic behavior of the type distribution for a large Λ, as described in Proposition 8, with the last expression (A.6) yields Equation A.2 in the statement. $\square$

The terms defining the illiquidity discount in (A.2) are rather intuitive. The fist term in curly brackets refers to the severity of the search friction. The second term in curly brackets refers to both the respective bargaining power of the agents bargaining and to the time it would take them to find another counter-party, should the negotiation collapse. The third term in curly brackets measures the risk sharing benefits that the investors are bargaining on.

Proofs for Section 1.3

Proof 38 (Proof of Proposition 3)**.** Let a be an agent with type $i_a\theta$ and wealth w_a. She met another agent b. Clearly, no trade will be possible unless the holdings of b are $\bar{\theta}$. I denote the two other characteristics of b by i_b and w_b.

There is a surplus for a and b to share if

$$
\emptyset \neq \left\{ \tilde{P} : \begin{array}{ll} V\left(w_a - \left(\bar{\theta} - \theta\right)\tilde{P}, i_a\bar{\theta}\right) \geq & V\left(w_a, i_a\theta\right) \\ V\left(w_b - \left(\theta - \bar{\theta}\right)\tilde{P}, i_b\theta\right) \geq & V\left(w_b, i_b\bar{\theta}\right) \end{array} \right\}.
$$

Under Assumption (2), this is equivalent to

$$
\emptyset \neq \mathscr{P} \triangleq \left\{ \tilde{P} : \; a(i_a\bar{\theta}) - a(i_a\theta) \geq \tilde{P}\left(\bar{\theta} - \theta\right) \geq a(i_b\bar{\theta}) - a(i_b\theta) \right\},
$$

or to

$$
a(i\bar{\theta}) - a(i\theta) + a(j\theta) - a(j\bar{\theta}) \geq 0.
$$

This proves the first two statements.

Now, if there actually is a surplus to share, the outcome of the bargaining is given by the Nash bargaining solution. Namely, a and b trade the asset at the price P_d so that

$$P_d = \arg\max_{\tilde{P}\in\mathscr{P}} \left(V(w_a - \tilde{P}(\bar{\theta}-\theta), i_a\bar{\theta}) - V(w_a, i_a\theta)\right)^{\eta_\theta} \cdot \\ \cdot \left(V(w_b - \tilde{P}(\theta-\bar{\theta}), i_b\theta) - V(w_b, i_b\bar{\theta})\right)^{1-\eta_\theta}.$$

Unless $\mathscr{P}$ is reduced to a single point, in which case the solution of the optimization is trivial, the first order condition characterize the point of maximum P_d as the solution to

$$\begin{aligned} &\eta_\theta \frac{\partial_w \left(V(w_a - P_d(\bar{\theta}-\theta), i_a\bar{\theta})\right)}{V(w_a + P_d(\bar{\theta}-\theta), i_a, \bar{\theta}) - V(w_a, i_a\theta)} \\ =&(1-\eta_\theta)\frac{\partial_w \left(V(w_b - P_d(\theta-\bar{\theta}), i_b\theta)\right)}{V(w_b - P_d(\theta-\bar{\theta}), i_b, \theta) - V(w_b, i_b\bar{\theta})} \end{aligned} \tag{A.7}$$

which, with Assumption 2, becomes (1.17). □

Proof 39 (Proof of Proposition 4)**.** The first order necessary condition for the maximization over the consumption rate is

$$\gamma e^{-\gamma c} - \frac{\partial V}{\partial w}(w, i\theta) = 0.$$

Recalling the Assumption 2, and solving for c yields a unique candidate $c(i\theta)$ which, by concavity of the objective function, is a point of maximum.

A similar argument yields the optimal liquid holdings $\pi(i\theta)$. □

Proof 40 (Proof of Proposition 5)**.** Starting from the HJB equation (1.13), picking a type $i\theta$, using Proposition (3) to transform the expected value into a deterministic quantity, Proposition 4 to express the optimal consumption, Proposition 3 to express the bagained price P_d, and normalizing by $r\gamma V(w, i, \theta)$, I obtain

$$\begin{aligned} 0 =& r - \rho - r\log(r) + r^2\gamma\bar{a} - r\gamma m_e \\ &+ \frac{\alpha}{r\gamma}\left(\frac{\alpha}{\gamma} - r\right) w \\ &+ r a(i\theta) - \kappa(i\theta) \\ &+ \lambda_{i\bar{i}} \frac{\left(e^{-r\gamma(a(\bar{i},\theta) - a(i,\theta))} - 1\right)}{-r\gamma} \\ &+ 2\Lambda\mu\left(\bar{i}\bar{\theta}\right)\left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}(a)\right)}{-r\gamma}\right]^+. \end{aligned} \tag{A.8}$$

Now, I can choose the constant

$$\bar{a} = \frac{1}{r\gamma}\left(-1 + \frac{\rho}{r} + \log(r) + \gamma m_e\right),$$

which sets the first line of the right hand side to zero. Also, as the equation above must hold for any value of the liquid holdings w,

$$\alpha\left(r - \frac{\alpha}{\gamma}\right) = 0.$$

Under Assumption 2, this requires $\alpha = r\gamma$ and the second line of the right hand side equals zero. Taking these two observations into account yields the system (1.20).

It remains to show that this equation admits exactly one solution. I split my argument into four steps.

Step 1 I first rearrange the four equations described in (1.20) into two. Namely, defining the variables

$$\Delta_\Theta \stackrel{\Delta}{=} a(1\Theta) - a(2\Theta) \tag{A.9}$$

and

$$\Delta_0 \stackrel{\Delta}{=} a(20) - a(10), \tag{A.10}$$

and taking the corresponding differences in the HJB equations (1.20) ensures that

$$\begin{aligned} 0 =& r\Delta_0 - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma\Delta_0} - 1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma\Delta_0} - 1}{-r\gamma} \\ &- 2\Lambda\left(\mu(1\Theta)\left[\frac{\chi\left(\eta_0, -\Delta_0 - \Delta_\Theta\right)}{-r\gamma}\right]^+ - \mu(2\Theta)\left[\frac{\chi\left(\eta_0, \Delta_0 + \Delta_\Theta\right)}{-r\gamma}\right]^+\right) \\ \stackrel{\Delta}{=}& F_0\left(\Delta_0, \Delta_\Theta\right) \end{aligned} \tag{A.11}$$

and

$$\begin{aligned} 0 =& r\Delta_\Theta - \kappa(1\Theta) + \kappa(2\Theta) - \lambda_{12}\frac{e^{r\gamma\Delta_\Theta} - 1}{-r\gamma} + \lambda_{21}\frac{e^{-r\gamma\Delta_\Theta} - 1}{-r\gamma} \\ &- 2\Lambda\left(\mu(20)\left[\frac{\chi\left(\eta_\Theta, -\Delta_0 - \Delta_\Theta\right)}{-r\gamma}\right]^+ - \mu(10)\left[\frac{\chi\left(\eta_\Theta, \Delta_0 + \Delta_\Theta\right)}{-r\gamma}\right]^+\right) \\ \stackrel{\Delta}{=}& F_\Theta\left(\Delta_0, \Delta_\Theta\right). \end{aligned} \tag{A.12}$$

Inspection ensures that, for any Δ_Θ, the function $F_0(\cdot, \Delta_\Theta)$ is strictly increasing with a range equal to the entire real line. It also ensures that, for any given Δ_0, the function $F_0(\Delta_0, \cdot)$ is strictly increasing with the bounded range

$$r\Delta_0 - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma\Delta_0}-1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma\Delta_0}-1}{-r\gamma} + \frac{2\Lambda}{r\gamma}\left[-\mu(1\Theta), \mu(2\Theta)\right]. \tag{A.13}$$

Similar properties hold for F_Θ.

Step 2 Given these properties of F_0 and F_Θ, I can define the functions

$$\Phi_0, \Phi_\Theta : \mathbb{R} \to \mathbb{R}$$

by requiring that, for any $x \in \mathbb{R}$,

$$0 = F_0(\Phi_0(x), x) = F_\Theta(x, \Phi_\Theta(x)). \tag{A.14}$$

The monotonicity properties also ensure that both Φ_0 and Φ_Θ are decreasing. I show two more properties of these functions.

First, these functions decrease relatively slowly. Namely, for any choice of $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$, it follows from (A.14) that

$$F_0\left(\Phi_0(x) - y, x + y\right) < F_0(\Phi_0(x), x) = 0 = F_0\left(\Phi_0(x+y), x+y\right),$$

meaning that

$$y + \Phi_0(x+y) - \Phi_0(x) > 0.$$

As a result, the function

$$x \mapsto x + \Phi_0(x) \tag{A.15}$$

is increasing and, by a similar argument, so is

$$x \mapsto x + \Phi_\Theta(x).$$

Second, their range is compact. Let me first consider Φ_Θ. Recalling the bounded range described by (A.13), I may write, for any pair $(\Delta_0, \Delta_\Theta) \in \mathbb{R}^2$,

$$F_0^L(\Delta_0) \leq F_0(\Delta_0, \Delta_\Theta) \leq F_0^U(\Delta_0),$$

where I defined

$$F_0^L(x) \triangleq rx - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma x}-1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma x}-1}{-r\gamma x} - \frac{2\Lambda}{r\gamma}\mu(1\Theta)$$

and

$$F_0^U(x) \triangleq rx - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma x}-1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma x}-1}{-r\gamma x} + \frac{2\Lambda}{r\gamma}\mu(2\Theta)$$

Inspection now ensures that, for $b_{U,0}$ large enough, $F_0^L\left(b_{U,0}\right) \geq 0$, and that for $b_{L,0}$ small enough, $F_0^U\left(b_{L,0}\right) \leq 0$. But then, for any Δ_Θ,

$$F_0(b_{L,0}, \Delta_\Theta) \leq F_0^U(b_{L,0}) \leq 0 \leq F_0^L(b_{U,0}) \leq F_0(b_{U,0}, \Delta_0).$$

Keeping the monotonicity and continuity of F_0 in mind, this implies that $\Phi_0\left(\Delta_\Theta\right) \in \left[b_{L,0}, b_{U,0}\right]$, and thus that

$$\Phi_0\left(\mathbb{R}\right) \subset \left[b_{L,0}, b_{U,0}\right].$$

A similar argument formulated with F_Θ would yield two other constants $b_{L,\Theta}$ and $b_{U,\Theta}$ so that

$$\Phi_\Theta\left(\mathbb{R}\right) \subset \left[b_{L,\Theta}, b_{U,\Theta}\right].$$

In particular, if I define

$$\Omega \triangleq \left[b_{L,\Theta} \wedge b_{L,0}, b_{U,0} \vee b_{U,0}\right],$$

then

$$\Phi\left(\Omega\right) \triangleq \left(\Phi_0, \Phi_\Theta\right)\left(\Omega\right) \subset \Omega \times \Omega.$$

Step 3 I now show that Φ is a contraction and, as a result, admits a unique fixed point. First, a direct verification shows the continuity of Φ_0 and Φ_Θ.

Second, if I choose $x \in \Omega$ so that

$$\Phi_0\left(x\right) + x \neq 0, \tag{A.16}$$

then, an application of the Implicit Function Theorem based on the relation (A.14) ensures that Φ_l is differentiable at x, with derivative given by

$$\Phi_0'(x) = -\frac{\frac{\partial F_0}{\partial \Delta_0}(\Phi_0(x), x)}{\frac{\partial F_0}{\partial \Delta_\Theta}(\Phi_0(x), x)}$$

$$= -\frac{2\Lambda\begin{pmatrix} \mathbf{1}_{\{-\Phi_0(x)-x>0\}}\mu(1\Theta)(-1)\frac{\partial \chi}{\partial \epsilon}\left(\eta_{20}, -\Phi_0(x) - x\right) \\ -\mathbf{1}_{\{\Phi_0(x)+x>0\}}\mu(2\Theta)\frac{\partial \chi}{\partial \epsilon}\left(\eta_{10}, \Phi_0(x) + x\right) \end{pmatrix}}{r + \lambda_{12}e^{r\gamma x} + \lambda_{21}e^{-r\gamma x} + 2\Lambda\begin{pmatrix} \mathbf{1}_{\{-\Phi_0(x)-x>0\}}\mu(1h\Theta)(-1)\frac{\partial \chi}{\partial \epsilon}\left(\eta_{20}, -\Phi_0(x) - x\right) \\ -\mathbf{1}_{\{\Phi_0(x)+x>0\}}\mu(2\Theta)\frac{\partial \chi}{\partial \epsilon}\left(\eta_{10}, \Phi_0(x) + x\right) \end{pmatrix}}.$$

Now, one checks that the numerator is positive, bounded on Ω, and also appears in the denominator. Further, the second part of the denominator,

$$r + \lambda_{12}e^{r\gamma x} + \lambda_{21}e^{-r\gamma x},$$

is also positive, and bounded on Ω. Then, there exists a constant $C \in (0,1)$, independent of the choice of x, and for which

$$\left|\Phi_0'(x)\right| < C.$$

on Ω.

Finally, remembering the monotonicity of the function in (A.15), there is at most one point in Ω where (A.16) does not hold and where, as a result, Φ_0 is not differentiable.

Summing up, the restriction of Φ_0 to Ω maps a compact into itself, is continuous, is differentiable everywhere but possibly at one point, and has a derivative whose absolute value that is bounded strictly below one.

This argument can be adapted for Φ_Θ, and classical contraction argument then ensures that Φ admits a unique fixed point $\left(\Delta_0^*, \Delta_\Theta^*\right)$ in Ω. Finally, as the range of Φ is already contained in Ω, this is the unique fixed point over $\mathbb{R}^2$.

Step 4 Finally, given the fixed point of Φ, the solution β to the HJB equations (1.20) can be recovered. For example,

$$a(1\Theta) = \frac{1}{r}\left(\kappa(1h) + \lambda_{12}\frac{e^{-r\gamma\Delta_h^*} - 1}{-r\gamma} + 2\Lambda\mu(20)\left[\chi\left(\eta_\Theta, -\Delta_0^* - \Delta_\Theta^*\right)\right]^-\right). \tag{A.17}$$

In particular, there is exactly one solution to the system of HJB equations (1.20). □

Proofs for Section 1.4

Proof 41 (Proof of Proposition 7)**.** The proposition and its proof are in Duffie et al. [2005]. I only give a partial sketch to introduce some notation.

There are three linear relations linking the components of a stationary distribution μ. They follow from the stationary distribution of endowment correlation types and from the market clearing condition (1.28), and are

$$\begin{cases} \mu(10) + \mu(1\Theta) & = \mu_1 \\ \mu(20) + \mu(2\Theta) & = \mu_2 \\ \mu(1\Theta) + \mu(2\Theta) & = \frac{S_d}{\Theta} \end{cases}. \tag{A.18}$$

One can then use these equations to express one of the flow conditions (1.25) as an equation in, say, $\mu(20)$ only. This yields the quadratic equation

$$0 = \mu(20)^2 + b\left(\frac{1}{\Lambda}\right)\mu(20) + c\left(\frac{1}{\Lambda}\right) \triangleq Q\left(\mu(2l), \frac{1}{\Lambda}\right), \tag{A.19}$$

where

$$\begin{aligned} b\left(\frac{1}{\Lambda}\right) &\triangleq \frac{S_d}{\Delta_\theta} - \mu_2 + \frac{1}{\Lambda}\frac{\lambda_{12}+\lambda_{21}}{2}, \\ c\left(\frac{1}{\Lambda}\right) &\triangleq -\frac{1}{\Lambda}\frac{\lambda_{12}}{2}\left(1 - \frac{S_d}{\Theta}\right). \end{aligned} \tag{A.20}$$

Solving this equation already characterize a unique candidate. □

I will use the following results when proving Corollary 12 and Proposition 8. They follow from the characterization (A.19).

Lemma 42. *The sensitivity of the stationary cross-sectional distribution of types to the illiquidity level satisfies*

$$\frac{\partial}{\partial\frac{1}{\Lambda}}\mu(1\Theta) = \frac{\partial}{\partial\frac{1}{\Lambda}}\mu(20) = -\frac{\partial}{\partial\frac{1}{\Lambda}}\mu(10) = -\frac{\partial}{\partial\frac{1}{\Lambda}}\mu(2\Theta) = \frac{-\mu(20)\frac{\lambda_{12}+\lambda_{21}}{2} + \frac{\lambda_{12}}{2}\left(1 - \frac{S_d}{\Theta}\right)}{\mu(1\Theta) + \mu(20) + \frac{1}{\Lambda}\frac{\lambda_{12}+\lambda_{21}}{2}}, \tag{A.21}$$

which is positive. Also,

$$\frac{\partial}{\partial\frac{1}{\Lambda}}\big(\lambda\mu(1\Theta)\mu(20)\big) = -\frac{1}{\left(\frac{1}{\Lambda}\right)^2}\frac{\mu(20)\mu(1\Theta)\frac{1}{2\Lambda}(\lambda_{12}+\lambda_{21})}{\mu(2l) + \mu(1h) + \frac{1}{\lambda}\frac{\lambda_{12}+\lambda_{21}}{2}} \tag{A.22}$$

and is negative. Finally, for $i\theta = 1\Theta, 2 0$*,*

$$\frac{\partial}{\partial \frac{1}{\Lambda}} \left(\lambda \mu(i\theta)\right)$$

and is negative as well.

Proof 43 (Proof of Lemma 42)**.** For the first statement, the sensitivity of $\mu(2 0)$ on the illiquidity level follows from an application of the Implicit Function Theorem. The relation between the various sensitivities then follows from (A.18).

Now, recalling Equations (1.25) and (1.28), I deduce from (A.21) that

$$\frac{\partial}{\partial \frac{1}{\Lambda}} \mu(1\Theta) = \frac{\partial}{\partial \frac{1}{\Lambda}} \mu(2 0) = \frac{\lambda \mu(1\Theta)\mu(2 0)}{\mu(1\Theta) + \mu(2 0) + \frac{1}{\Lambda} \frac{\lambda_{12} + \lambda_{21}}{2}},$$

which is positive. A direct calculation then yields (A.22). Finally the last sensitivity follows from the elementary observation that, if the product of two positive functions is increasing, and if the first term in the product is decreasing, then the second one must be increasing. $\square$

Proof 44 (Proof of Proposition 8)**.** From the proof of Proposition 7,

$$\mu\left(2 0, \frac{1}{\Lambda}\right) = \frac{1}{2}\left(-b\left(\frac{1}{\Lambda}\right) + \sqrt{\left(b\left(\frac{1}{\Lambda}\right)\right)^2 - 4c\left(\frac{1}{\Lambda}\right)}\right).$$

Now, as

$$\lim_{\frac{1}{\Lambda} \to 0} b\left(\frac{1}{\Lambda}\right) = \frac{S_d}{\Theta} - \mu_2,$$

which I assumed to be negative, and

$$\lim_{\frac{1}{\Lambda} \to 0} c\left(\frac{1}{\Lambda}\right) = 0,$$

it follows that

$$\lim_{\frac{1}{\Lambda} \to 0} \mu\left(2 0, \frac{1}{\Lambda}\right) = \mu_2 - \frac{S_d}{\Theta}.$$

Recalling the linear relationships (A.18), this yields the asymptotic distribution.

Now, using the previous lemma yields

$$\partial_{\frac{1}{\Lambda}}\mu(20) = \frac{-\mu(20)\frac{\lambda_{12}+\lambda_{21}}{2} + \frac{\lambda_{12}}{2}\left(1-\frac{S_d}{\Theta}\right)}{2\mu(20)+\frac{S_d}{\Theta}-\mu_2+\frac{1}{\Lambda}\frac{\lambda_{12}+\lambda_{21}}{2}} \xrightarrow[]{\frac{1}{\Lambda}\to 0} \frac{\lambda_{21}}{2}\frac{\frac{S_d}{\Theta}}{\mu_2-\frac{S_d}{\Theta}},$$

which concludes. □

Proof 45 (Proof of Proposition 10)**.** Keeping Proposition 4 in mind, the equilibrium condition for the centralized market becomes

$$S_c = \mathrm{E}^{\mu(i\theta)}\left[\pi(i\theta)\right] = \frac{1}{\sigma_c^2}\left(\frac{1}{r\gamma}(m_c - rP_c) - \mathrm{E}^{\mu(i\theta)}\left[\Sigma_{ic}\right] - \Sigma_{cd}\,\mathrm{E}^{\mu(i\theta)}\left[\theta\right]\right).$$

Now, realizing that

$$\mathrm{E}^{\mu(i\theta)}\left[\Sigma_{ic}\right] = \mu_1\Sigma_{1c} + \mu_2\Sigma_{2c} \overset{(\Delta)}{=} \Sigma_{\eta c}$$

is independent of the trading on the OTC market, and that, thanks to the market clearing condition (1.28), so is

$$\mathrm{E}^{\mu(i\theta)}\left[\theta\right] = S_d,$$

I can already solve for the equilibrium price, which yields (1.34). Then, combining this last result and the characterization (1.18) of the optimal liquid holdings yields the expression (1.35) in the statement.

I now turn to the OTC market. The existence and uniqueness follows from two elementary observations. First, the value function of a, say, 1h-agents is only impacted by μ via $\mu(20)$ and only as long as $\epsilon_{1\Theta}(a) > 0$. Otherwise, 1h-agents have no intention to trade and, as a result, no interest in knowing how often a counter-party may be met. In mathematical terms, this reads

$$\begin{aligned}
\mu(a,20)\left[\frac{\chi(\eta_0,\epsilon_{1\Theta}(a))}{-r\gamma}\right]^+ &= \mu(a,20)\mathbf{1}_{\{\epsilon_{1\Theta}(a)>0\}}\frac{\chi\left(\eta_0,\epsilon_{1\Theta}(a)\right)}{-r\gamma} \\
&= \mu^{1\Theta\to 20}(20)\mathbf{1}_{\{\epsilon_{1\Theta}(a)>0\}}\frac{\chi\left(\eta_0,\epsilon_{1\Theta}(a)\right)}{-r\gamma} \\
&= \mu^{1\Theta\to 20}(20)\left[\frac{\chi\left(\eta_0,\epsilon_{1\Theta}(a)\right)}{-r\gamma}\right]^+
\end{aligned}$$

In particular, this means that I can choose

$$\hat{\mu} = \begin{pmatrix} \mu^{2\Theta\to 10}(10) \\ \mu^{1\Theta\to 20}(1\Theta) \\ \mu^{1\Theta\to 20}(20) \\ \mu^{2\Theta\to 10}(2\Theta) \end{pmatrix}$$

as the "density" in (1.20). Note that this vector does not depend on a but does not define a density any more.

The second observation is that the proof of Proposition 5 remains valid when the components of μ are only positive numbers, and do not necessarily sum up to one. As a result, there is exactly one solution to the HJB equations defining an equilibrium, which shows the uniqueness and existence of an equilibrium.

I must still characterize the ordering of the valuations of the illiquid asset d or, equivalently, characterize the trading pattern on the OTC market. To do so I first characterize the ordering when the OTC market becomes arbitrarily liquid, and then show that this ordering is maintained at any illiquidity level. The actual argument is articulated around three claims.

Claim 1 I first show that an equilibrium a of the model can be bounded by constants that are independent of the illiquidity level.

Proof of Claim 1. Let $\{\Lambda_n\}_{n\geq 0}$ be a sequence of intensities be given, and let $\{a_n\}_{n\geq 0}$ be the corresponding sequence of equilibria. Let me assume, for the sake of contradiction, that there is an agent type $i\theta$ for which the sequence $\{a_n(i\theta)\}_{n\geq 0}$ is unbounded. I first assume it is unbounded below, meaning that, maybe up to taking a subsequence,

$$\lim_{n\to\infty} a_n(i\theta) = -\infty, \tag{A.23}$$

Recalling the HJB equations (1.20) and the first part of this proof,

$$-\lambda_{i\bar{i}} \frac{e^{-r\gamma(a_n(\bar{i}\theta)-a_n(i\theta))}-1}{-r\gamma} = -r a_n(i\theta) + \kappa(i\theta) + 2\Lambda_n \hat{\mu}\left(\Lambda_n, \bar{i}\bar{\theta}\right) \left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}(a)\right)}{r\gamma}\right]^+. \tag{A.24}$$

But, recalling (A.23), the left hand side of (A.24) is bounded below by a sequence that grows arbitrarily. As a result,

$$\lim_{n\to\infty} a_n(i\theta) - a_n(\bar{i}\theta) = +\infty. \tag{A.25}$$

and, recalling (A.23) one more time,

$$\lim_{n\to\infty} a_n(\bar{i}\theta) = -\infty. \tag{A.26}$$

Now, if (A.25) follows from (A.23), from (A.26) I can conclude that

$$\lim_{n\to\infty} a_n(\bar{i}\theta) - a_n(i\theta) = +\infty. \tag{A.27}$$

In particular, both (A.25) and (A.27) follow from (A.23), which is impossible. There is thus no sequence of equilibria that is unbounded below.

It remains to see whether a sequence of equilibria can be unbounded above. Let me assume that, maybe choosing a subsequence,

$$\lim_{n\to\infty} a_n(1\Theta) = +\infty. \tag{A.28}$$

Choosing the type 1Θ is without loss of generality. Before pursuing the argument I note that, assuming an agent of type $i\theta$ does not trade in equilibrium, it follows from (1.20) and the first part of the proof that

$$\begin{aligned}
0 =& r a_n(i\theta) - \kappa(i\theta) - \lambda_{i\bar{i}} \frac{e^{-r\gamma\left(a_n(\bar{i}\theta)-a_n(i\theta)\right)}-1}{-r\gamma} - 2\Lambda_n \hat{\mu}(\lambda, \bar{i}\bar{\theta}) \left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}(a)\right)}{-r\gamma}\right]^+ \\
=& r a_n(i\theta) - \kappa(i\theta) - \lambda_{i\bar{i}} \frac{e^{-r\gamma\left(a_n(i\theta)-a_n(\bar{i}\theta)\right)}-1}{-r\gamma} \\
\geq& r a_n(i\theta) - \kappa(i\theta) - \lambda_{i\bar{i}} \frac{1}{r\gamma}.
\end{aligned}$$

In other words, I have an a priori upper bound on $a_n(i\theta)$. Namely,

$$\frac{1}{r}\left(\kappa(i\theta) + \frac{\lambda_{i\bar{i}}}{r\gamma}\right) \geq a_n(i\theta). \tag{A.29}$$

Now, two further cases must be distinguished, depending on whether 1Θ-agents are willing to trade or not. Maybe choosing a further subsequence, I assume that 1Θ-agents never trade. In this case, combining (A.29) and (A.28) yields

$$\frac{1}{r}\left(\kappa(1\Theta) + \frac{\lambda_{12}}{r\gamma}\right) \geq \lim_{n\to\infty} a_n(1\Theta) = +\infty,$$

which is a contradiction. The only possibility left is thus for the agents with type 1Θ are willing to trade. I can thus assume that, for any $n \geq 0$,

$$a_n(10) - a_n(1\Theta) - a_n(20) + a_n(2\Theta) \geq 0.$$

Using (A.29) for the two types of agent that do not trade, meaning 10 and 2Θ, then yields

$$\frac{1}{r}\left(\kappa(10)+\kappa(2\Theta)+\frac{\lambda_{12}}{r\gamma}+\frac{\lambda_{21}}{r\gamma}\right)\geq a_n(1\Theta)+a_n(20).$$

From this last inequality and (A.28) I deduce that

$$\lim_{n\to\infty} a_n(20) = -\infty,$$

which will, by the first part of this proof, lead to a contradiction.

To sum up, there are no circumstances under which an unbounded sequence of equilibria can be found. □

This first claim is needed when proving the second one.

Claim 2 For a sufficiently large meeting intensity Λ, the corresponding equilibrium $a(\Lambda)$ satisfies

$$\epsilon_{1\Theta}(a) > 0$$

exactly when $\mathscr{S} > 0$, with $\mathscr{S}$ defined in (1.24).

Proof of Claim 2. Let me choose a sequence $\{\Lambda_n\}_{n\geq 0}$ of meeting intensities so that

$$\lim_{n\to\infty} \Lambda_n = +\infty.$$

By Claim 1, there exists two constants $L < U$ so that

$$\forall n:\ a_n \in [L,U]^4 \tag{A.30}$$

I can thus choose a convergent subsequence, and call the limit a_∞. Maybe choosing a further subsequence, I assume that

$$\forall n: \epsilon_{1\Theta}(a_n) \overset{(\Delta)}{=} a_n(2\Theta) - a_n(20) + a_n(10) - a_n(1\Theta) \geq 0. \tag{A.31}$$

In other words, all along the sequence of intensities, and in the limit, agents with endowment correlations type 2 have the high valuation of the illiquid asset.

Under this assumption the HJB equations defining a_n become

$$\left\{\begin{array}{lclll} ra_n(10) & = & \kappa(10) & +\lambda_{12}\frac{e^{-r\gamma(a_n(20)-a_n(10))}-1}{-r\gamma} & \\ ra_n(1\Theta) & = & \kappa(1\Theta) & +\lambda_{12}\frac{e^{-r\gamma(a_n(2\Theta)-a_n(1\Theta))}-1}{-r\gamma} & +2\Lambda_n\hat{\mu}(\Lambda_n,20)\frac{\chi(\eta_\Theta,\epsilon_{1\Theta}(a))}{-r\gamma} \\ ra_n(20) & = & \kappa(20) & +\lambda_{21}\frac{e^{-r\gamma(a_n(10)-a_n(20))}-1}{-r\gamma} & +2\Lambda_n\hat{\mu}(\Lambda_n,1\Theta)\frac{\chi(\eta_0,\epsilon_{20}(a))}{-r\gamma} \\ ra_n(2\Theta) & = & \kappa(2\Theta) & +\lambda_{21}\frac{e^{-r\gamma(a_n(1\Theta)-a_n(2\Theta))}-1}{-r\gamma} & \end{array}\right. . \tag{A.32}$$

At this stage, I will consider the asymptotic behavior of the stationary type distribution, which requires to distinguish two cases.

I first assume

$$\mu_2 - \frac{S_d}{\Delta_\theta} > 0, \tag{A.33}$$

meaning that the marginal buyer of the illiquid asset has a high valuation. In this case, it is known from Lemma 8 that

$$\lim_{n\to\infty}\hat{\mu}(\Lambda_n,20) = \mu_2 - \frac{S_d}{\Theta} > 0.$$

As a result,

$$\lim_{n\to\infty}\Lambda_n\hat{\mu}(\Lambda_n,20) = \infty. \tag{A.34}$$

Now, as stated in (A.30) shows that the equilibria are bounded. Hence, (A.32) is only compatible with (A.34) if

$$\lim_{n\to\infty}\chi\left(\eta_\Theta,\epsilon_{1\Theta}(a)\right) = 0.$$

Recalling the definition of "χ" in (1.23), this is equivalent to

$$a_\infty(10) - a_\infty(1\Theta) = a_\infty(20) - a_\infty(2\Theta). \tag{A.35}$$

But then, as Lemma 8 ensures that

$$\lim_{n\to\infty}\Lambda_n\hat{\mu}(\Lambda_n,1\Theta) = \frac{\lambda_{12}}{2}\frac{\frac{S_d}{\Theta}}{\mu_2 - \frac{S_d}{\Theta}}$$

letting n go to $+\infty$ in (A.32) yields

$$\begin{cases} ra_\infty(10) &= \kappa(10) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(20)-a_\infty(10))}-1}{-r\gamma} \\ ra_\infty(1\Theta) &= \kappa(1\Theta) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(2\Theta)-a_\infty(1\Theta))}-1}{-r\gamma} \\ & & +\lim_{n\to\infty} 2\Lambda_n\hat{\mu}(\Lambda_n,20)\frac{\chi(\eta_\Theta,\epsilon_{1\Theta}(a))}{-r\gamma} \\ ra_\infty(20) &= \kappa(20) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(10)-a_\infty(20))}-1}{-r\gamma} \\ ra_\infty(2\Theta) &= \kappa(2\Theta) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(1\Theta)-a_\infty(2\Theta))}-1}{-r\gamma} \end{cases}. \tag{A.36}$$

Now, subtracting the second and third equations from the sum of the first and fourth ones in (A.36), and then repeatedly using (A.35), yields

$$\kappa(10)-\kappa(1\Theta)+\kappa(2\Theta)-\kappa(20) = \lim_{n\to\infty} 2\Lambda_n\hat{\mu}(\Lambda_n,20)\frac{\chi\left(\eta_\Theta,\epsilon_{1\Theta}(a)\right)}{-r\gamma}. \tag{A.37}$$

I draw two conclusions from this last equality. First, combining it with (A.38) yields

$$\begin{cases} ra_\infty(10) &= \kappa(10) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(20)-a_\infty(10))}-1}{-r\gamma} \\ ra_\infty(1\Theta) &= \kappa(10)+\kappa(2\Theta)-\kappa(20) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(2\Theta)-a_\infty(1\Theta))}-1}{-r\gamma} \\ ra_\infty(20) &= \kappa(20) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(10)-a_\infty(20))}-1}{-r\gamma} \\ ra_\infty(2\Theta) &= \kappa(2\Theta) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(1\Theta)-a_\infty(2\Theta))}-1}{-r\gamma} \end{cases}. \tag{A.38}$$

This system defines a contraction, as Proposition 53 below formally shows, which ensures the uniqueness of the asymptotic equilibrium β_∞.

Second, (A.37) is only compatible with the assumption (A.31) as long as

$$\mathscr{S} \overset{(\Delta)}{=} \kappa(10)-\kappa(1\Theta)-\kappa(20)+\kappa(2\Theta) \geq 0. \tag{A.39}$$

The case of

$$\frac{S_d}{\Theta} > \mu_2$$

is handled similarly.

Assuming the reverse inequality in (A.31) would also give a unique candidate for a_∞, but this time require that (A.39) also holds with a reverse inequality.

Summing up, if (A.39) holds, then the sequence of equilibria converges and, for n large enough, $\epsilon_{1\Theta}(a_n) > 0$. Otherwise, the sequence converges as well but, for n large enough, $\epsilon_{2\Theta}(a_n) < 0$. □

I have now characterized which trades are implemented when the meeting intensity is sufficiently large. The last step is to show that the trading pattern cannot be reverted by an increasing illiquidity level.

Claim 3 The surplus to be shared in bilateral trades is differentiable and decreasing in the meeting intensity. In other words, if $\epsilon_{i\theta}\left(a(\Lambda)\right) > 0$,

$$\frac{\partial}{\partial \Lambda}\epsilon_{i\theta}\left(a(\Lambda)\right) < 0.$$

In particular, the derivative exists.

Proof of Claim 3. Without loss of generality, I assume

$$\epsilon_{1\Theta}(a) \overset{(\Delta)}{=} a(10) - a(1\Theta) - a(20) + a(2\Theta) \overset{(\Delta)}{=} -\Delta_\Theta - \Delta_0 > 0, \tag{A.40}$$

meaning that the 2-agents have the high valuation. From the proof of Proposition 5, I know that for any given Λ, the pair $\Delta \overset{\Delta}{=} (\Delta_\Theta, \Delta_0)$ is the unique solution to the system

$$0 = F(\Delta;\Lambda) \overset{(\Delta)}{\Leftrightarrow} \begin{cases} 0 = & F_\Theta\left(\Delta_\Theta, \Delta_0; \Lambda\right) \\ 0 = & F_0\left(\Delta_0, \Delta_\Theta; \Lambda\right) \end{cases},$$

where the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ is implicitly defined in the last equation. Now, under the above assumption regarding the high valuation agents, I can write

$$\begin{aligned}
&\det\left(D_\Delta F\left(\Delta, \Lambda\right)\right) \\
&= \det \left(\begin{array}{c:c}
\begin{matrix} r + \lambda_{12} e^{r\gamma\Delta_h} + \lambda_{21} e^{-r\gamma\Delta_h} \\ +2\Lambda\mu(20)\frac{\frac{\partial\chi}{\partial\epsilon}\left(\eta_{1h}, -\Delta_l - \Delta_h\right)}{-r\gamma} \end{matrix} & \frac{2\Lambda\mu(20)\frac{\partial\chi}{\partial\epsilon}\left(\eta_{1h}, -\Delta_l - \Delta_h\right)}{-r\gamma} \\ \hdashline
\frac{2\Lambda\mu(1\Theta)\frac{\partial\chi}{\partial\epsilon}\left(\eta_{2l}, -\Delta_l - \Delta_h\right)}{-r\gamma} & \begin{matrix} r + \lambda_{21} e^{r\gamma\Delta_l} + \lambda_{12} e^{-r\gamma\Delta_l} \\ +\frac{2\Lambda\mu(1\Theta)\frac{\partial\chi}{\partial\epsilon}\left(\eta_{2l}, -\Delta_l - \Delta_h\right)}{-r\gamma} \end{matrix}
\end{array}\right) \\
&= \left(r + \lambda_{12} e^{r\gamma\Delta_h} + \lambda_{21} e^{-r\gamma\Delta_h}\right)\left(r + \lambda_{21} e^{r\gamma\Delta_l} + \lambda_{12} e^{-r\gamma\Delta_l}\right) \\
&\quad + \left(r + \lambda_{12} e^{r\gamma\Delta_h} + \lambda_{21} e^{-r\gamma\Delta_h}\right)\frac{2\Lambda\mu(1\Theta)\frac{\partial\chi}{\partial\epsilon}\left(\eta_{2l}, -\Delta_l - \Delta_h\right)}{-r\gamma} \\
&\quad + \left(r + \lambda_{21} e^{r\gamma\Delta_l} + \lambda_{12} e^{-r\gamma\Delta_l}\right)\frac{2\Lambda\mu(20)\frac{\partial\chi}{\partial\epsilon}\left(\eta_{1h}, -\Delta_l - \Delta_h\right)}{-r\gamma}.
\end{aligned} \tag{A.41}$$

Recalling from the definition (1.23) that χ is decreasing in its second argument, this last quantity is positive, which justifies an application of the Implicit Function Theorem. This ensures that Δ is, locally, a differentiable function $\Delta(\lambda)$ of the meeting intensity, with derivative

$$\begin{aligned}\partial_\lambda \Delta(\lambda) &= -\left(D_\Delta F(\Delta,\lambda)\right)^{-1} D_\lambda F(\Delta,\lambda)\\ &= \frac{-1}{\det\left(D_\Delta F\right)}\begin{pmatrix} \frac{\partial F_0}{\partial \Delta_0} & -\frac{\partial F_\Theta}{\partial \Delta_0} \\ -\frac{\partial F_0}{\partial \Delta_\Theta} & \frac{\partial F_\Theta}{\partial \Delta_\Theta}\end{pmatrix}\begin{pmatrix} \frac{\partial F_\Theta}{\partial \lambda} \\ \frac{\partial F_0}{\partial \lambda}\end{pmatrix}.\end{aligned}$$

But then,

$$\begin{aligned}&\frac{\partial}{\partial \Lambda}\left(\Delta_0+\Delta_\Theta\right)\\ =&\frac{-1}{\det\left(D_\Delta F\right)}\left(\left(\frac{\partial F_0}{\partial \Delta_0}-\frac{\partial F_0}{\partial \Delta_\Theta}\right)\frac{\partial F_\Theta}{\partial \lambda}+\left(\frac{\partial F_\Theta}{\partial \Delta_\Theta}-\frac{\partial F_\Theta}{\partial \Delta_0}\right)\frac{\partial F_\Theta}{\partial \Lambda}\right)\\ =&\frac{-1}{\det\left(D_\Delta F(\Delta,\Lambda)\right)}\begin{pmatrix} \left(r+\lambda_{21}e^{r\gamma\Delta_0}+\lambda_{12}e^{-r\gamma\Delta_0}\right)\frac{\chi(\eta_\Theta,-\Delta_0-\Delta_\Theta)}{-r\gamma}2\partial_\Lambda\left(\Lambda\mu(\Lambda,2l)\right) \\ +\left(r+\lambda_{12}e^{r\gamma\Delta_\Theta}+\lambda_{21}e^{-r\gamma\Delta_\Theta}\right)\frac{\chi(\eta_{2l},-\Delta_0-\Delta_\Theta)}{-r\gamma}2\partial_\Lambda\left(\Lambda\mu(\Lambda,1\Theta)\right)\end{pmatrix}.\end{aligned}$$

With (A.40), both $\chi\left(\eta_\Theta,-\Delta_0-\Delta_\Theta\right)$ and $\chi\left(\eta_0,-\Delta_0-\Delta_\Theta\right)$ are negative, As a result,

$$\partial_\Lambda\left(\Delta_\Theta(\Lambda)+\Delta_0(\Lambda)\right)>0$$

or, equivalently,

$$\partial_\Lambda\left(-\Delta_\Theta(\Lambda)-\Delta_0(\Lambda)\right)<0$$

which proves the claim. □

I can finally conclude the proof of Proposition 10. Indeed, assuming that $\mathscr{S}>0$, Claim 2 ensures that, if the meeting intensity Λ is larger than a certain threshold $\bar{\Lambda}$, then, $\epsilon_{1\Theta}\left(a(\Lambda)\right)>0$, meaning that 2-agents have the high valuation. But then, Claim 3 ensures that decreasing Λ increases $\epsilon_{1\Theta}\left(a(\Lambda)\right)$. In particular, 2-agents still have the high valuation for any value of the meeting intensity. The case where $\mathscr{S}<0$ is similar.

Proof 46 (Proof of Proposition 12)**.** Without loss of generality, I assume that 2-agents have the high valuation of the illiquid asset.

Regarding the OTC market, as the transaction size is fixed, the trading volume is proportional to

$$2\Lambda\mu(1\Theta)\mu(20),$$

meaning to the meeting intensity between 1Θ and 20 agents. From Lemma 42 this quantity is increasing in the meeting intensity.

Recalling the expressions (1.35) for the liquid holdings in equilibrium, the volume exchanged on the centralized market per unit of time is thus

$$\begin{aligned}
\mathrm{Vol} &= \frac{1}{2}\left\{\begin{array}{rl}
\lambda_{12}\mu(10) & |\pi(10)-\pi(20)| \\
+\lambda_{12}\mu(1\Theta) & |\pi(1\Theta)-\pi(2\Theta)| \\
+2\Lambda\mu(1\Theta)\mu(20) & |\pi(1\Theta)-\pi(10)| \\
+2\Lambda\mu(1\Theta)\mu(20) & |\pi(20)-\pi(2\Theta)| \\
+\lambda_{21}\mu(20) & |\pi(20)-\pi(10)| \\
+\lambda_{21}\mu(2\Theta) & |\pi(2\Theta)-\pi(1\Theta)|
\end{array}\right\}. \\
&= \frac{1}{2\Sigma_{cc}}\left\{\left(\lambda_{12}\mu_1+\lambda_{21}\mu_2\right)|\Sigma_{1c}-\Sigma_{2c}|+4\Lambda\mu(1\Theta)\mu(20)\,|\Sigma_{cd}|\,\Theta\right\} \\
&= \frac{1}{\Sigma_{cc}}\left\{\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}}|\Sigma_{1c}-\Sigma_{2c}|+2\Lambda\mu(1\Theta)\mu(20)\,|\Sigma_{cd}|\,\Theta\right\}
\end{aligned} \tag{A.42}$$

Lemma 42 shows that the trading volume in increasing in Λ. □

Proof 47 (Proof of Proposition 13)**.** To obtain a model without OTC market, we can set $\Theta = 0$. In this case Equation (A.42) in the last proof immediately shows that the trading volume in c drops.

Further, letting the meeting intensity Λ grow arbitrarily in Equation (A.42), and recalling Proposition 8, I calculate the asymptotic level of trading in c as

$$\lim_{\Lambda\to\infty}\mathrm{Vol}(\Lambda) = \frac{1}{\Sigma_{cc}}\left\{\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}}|\Sigma_{1c}-\Sigma_{2c}|+2\lambda_{21}S_d\,|\Sigma_{cd}|\right\}.$$

Now, in a Walrasian setting, the market for d can only clear if the investors with a high valuation randomize their decision to buy the asset d. If the inequality (1.36) holds, inspection shows that the trading volume on the market for c in a Walrasian setting is

$$\begin{aligned}
\mathrm{Vol}_W &= \frac{1}{2}\left(\begin{array}{l}
\mu_1\lambda_{12}\frac{\mu_2-\frac{S_d}{\Theta}}{\mu_2}\left|\mu(20)-\pi(10)\right| \\
+\mu_1\lambda_{12}\frac{\frac{S_d}{\Theta}}{\mu_2}\left|\mu(2\Theta)-\pi(10)\right| \\
+\frac{S_d}{\Theta}\lambda_{21}\,|\pi(10)-\pi(2\Theta)| \\
+\left(\mu_2-\frac{S_d}{\Theta}\right)\lambda_{21}\,|\pi(10)-\pi(20)|
\end{array}\right) \\
&= \left(\begin{array}{l}
\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}}\left|\mu(20)-\pi(10)\right| \\
+\frac{S_d}{\Theta}\lambda_{21}\left(\left|\mu(2\Theta)-\pi(10)\right|-\left|\mu(20)-\pi(10)\right|\right)
\end{array}\right).
\end{aligned}$$

The triangular inequality ensures that

$$\left|\mu(2\Theta)-\pi(10)\right|-\left|\mu(20)-\pi(10)\right| \le \left|\mu(2\Theta)-\pi(20)\right|$$

and, as a result, that

$$\mathrm{Vol}_W \leq \lim_{\Lambda \to \infty} \mathrm{Vol}(\Lambda).$$

Inspection finally shows that Inequality (1.36) defines the cases for which the triangular inequality is strict. □

Proof 48 (Proof of Proposition 16)**.** The optimization for the optimal design of the liquid asset is

$$\max_{(a_c,b_c)} f(a_c,b_c) \overset{(\Delta)}{=} \max_{(a_c,b_c)} \{|w_1 a_c + w_2 b_c| + |w_3 a_c + w_4 b_c|\} \tag{A.43}$$

under the conditions

$$\|(a_c,b_c)\|_2 = 1, \tag{A.44}$$

$$\det\left(\left(\begin{array}{c:c} a_d & a_c \\ b_d & b_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c:c} a_1 - a_2 & \alpha_c \\ b_1 - b_2 & \beta_c \end{array}\right)\right) > 0. \tag{A.45}$$

I characterize the solution to this problem by maximizing f under the constraint (A.44) and making sure that the constraint (A.45) holds *ex post*.

A solution to the maximization of f under (A.44) exists because f is continuous and the optimization domain is compact.

The objective function f is piecewise linear and I define

$$\mathscr{A} \overset{\Delta}{=} \left\{(x,y) \in \mathbb{R}^2 : \; w_1 x + w_2 y \geq 0\right\}$$

and

$$\mathscr{B} \overset{\Delta}{=} \left\{(x,y) \in \mathbb{R}^2 : \; w_3 x + w_4 y \geq 0\right\}$$

to describe this piecewise structure. Clearly, the optimal risk-profile (a_c, b_c) belongs either to $\mathscr{D}_1 \overset{\Delta}{=} (A \cap B^c) \cup (A^c \cap B)$ or to $\mathscr{D}_2 \overset{\Delta}{=} (A \cap B) \cup (A^c \cap B^c)$. In the first case, the method of Lagrange multipliers characterizes the optimal risk profile as

$$\left\{\begin{array}{rcl} \nabla_{(a_c,b_c)} f\big|_{\mathscr{D}_1}(a_c,b_c) & = & L\nabla_{(a_c,b_c)}\left(\|(a_c,b_c)\|_2\right) \\ \|(a_c,b_c)\|_2 & = & 1 \end{array}\right. ,$$

with $L \in \mathbb{R}$ being the Lagrange multiplier. Solving this system for (a_c, b_c) yields the unique candidate

$$\begin{pmatrix} a_c \\ a_d \end{pmatrix} = \pm\frac{1}{\nu}\begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}, \tag{A.46}$$

with the constant

$$\nu \overset{\Delta}{=} \sqrt{(w_1 - w_3)^2 + (w_2 - w_4)^2}$$

ensuring the normalization $\Sigma_{cc} = 1$.[3] I must still make two checks to ensure the validity of this candidate as a solution to the original problem. First, does the candidate satisfy the constraint (A.45)? Plugging (A.46) into (A.45) yields

$$\frac{2\Theta\Lambda\lambda_{12}\lambda_{21}\mu(20)\mu(1\Theta)\left((b_2 - b_1)\,a_d + (a_1 - a_2)\,b_d\right)^2}{\nu^2\left(\lambda_{1,2} + \lambda_{2,1}\right)} > 0$$

and the consistency constraint is necessarily satisfied. Second, does the candidate actually belongs to $\mathscr{D}_1$? A vector $\left(\tilde{a}_c, \tilde{b}_c\right)$ belongs to $\mathscr{D}_1$ exactly when

$$(w_1 a_c + w_2 b_c)(w_3 a_c + w_4 b_c) < 0.$$

For the choice $(a_c, b_c) = \left(\tilde{a}_c, \tilde{b}_c\right)$, this inequality becomes the assumption (1.55) in the statement, and is thus satisfied.

Finally, we can consider the second case $(a_c, b_c) \in \mathscr{D}_2$ and follow the same steps as in the first case. In this second case, however, the unique candidate does not satisfy the condition (A.45) and there is no solution to the original problem (A.43). □

Lemma 49. *The equilibrium price P_c of the liquid asset and the corresponding holdings $\{\pi(i\theta; m_2, t\}_{i\theta}$ are uniquely defined in the asymptotic case characterized by Equation* (1.47) *and Equation* (1.48). *In case of an aggregate shock from the state (m_2, t) to $\tilde{m}_2, 0$, the price P_c jumps up when*

$$(\Sigma_{2c} - \Sigma_{1c})\left(\tilde{m}_2 - \mu_2(m_2, t)\right) < 0$$

and down when the other inequality holds. Finally, the "$\kappa(i\theta; m_2, t)$" defined in (1.45) *satisfy*

$$\begin{aligned}\mathscr{S} &\overset{\Delta}{=} \kappa(10; m_2, t) - \kappa(1\Theta; m_2, t) - \kappa(20; m_2, t) + \kappa(2\Theta; m_2, t) \\ &= \frac{r\gamma\Theta}{\Sigma_{cc}}\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right)\cdot\det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right) + o(\gamma)\end{aligned}$$

for any state (m_2, t) of the economy.

[3] The ± follows from the symmetry of the optimization, as discussed in the main text.

Proof. I start from the optimization over the liquid holdings $\tilde{\pi}$ in the HJB equation 1.44. The first-order condition for this optimization characterizes the optimal holdings $\pi(i\theta; m_2, t)$ as the unique solution to the equation

$$\begin{aligned} 0 &= \dot{P}_c(m_2,t) + m_c - rP_c(m_2,t) - r\gamma \begin{pmatrix} \Sigma_{ic} & \Sigma_{cd} & \Sigma_{cc} \end{pmatrix} \begin{pmatrix} 1 \\ \theta \\ \pi(i\theta; m_2, t) \end{pmatrix} \\ &+ \lambda_a \mathrm{E}^{M(\tilde{m}_2)} \left[\begin{array}{l} (P_c(\tilde{m}_2, 0) - P_c(m_2,t)) \cdot \\ \cdot \left(\begin{array}{l} \delta(i; m_2, t; \tilde{m}_2) \cdot \\ \cdot e^{-r\gamma(a(\bar{i}\theta;\tilde{m}_2,0)+\pi(i\theta;m_2,t)(P_c(\tilde{m}_2,0)-P_c(m_2,t))-a(i\theta;m_2,t))} \\ + (1-\delta(i; m_2, t; \tilde{m}_2)) \cdot \\ \cdot e^{-r\gamma(a(i\theta;\tilde{m}_2,0)+\pi(i\theta;m_2,t)(P_c(\tilde{m}_2,0)-P_c(m_2,t))-a(i\theta;m_2,t))} \end{array} \right) \end{array} \right] \end{aligned} \tag{A.47}$$

In the asymptotic case of a small risk-aversion to the jump risks, as characterized by the equations (1.47) and (1.48), this first-order condition becomes

$$\begin{aligned} 0 &= \dot{P}_c(m_2,t) + m_c - rP_c(m_2,t) - r\gamma \begin{pmatrix} \Sigma_{ic} & \Sigma_{cd} & \Sigma_{cc} \end{pmatrix} \begin{pmatrix} 1 \\ \theta \\ \pi(i\theta; m_2, t) \end{pmatrix} \\ &+ \lambda_a \left(\mathrm{E}^{M(\tilde{m}_2)} \left[P_c(\tilde{m}_2, 0) \right] - P_c(m_2,t) \right) + \mathcal{O}(\gamma). \end{aligned} \tag{A.48}$$

Equation (A.48) can be solved for $\pi(i\theta; m_2, t)$ in closed-form. A direct calculation then shows

$$\begin{aligned} \mathscr{S} &\triangleq \kappa(10; m_2, t) - \kappa(1\Theta; m_2, t) - \kappa(20; m_2, t) + \kappa(2\Theta; m_2, t) \\ &= \frac{r\gamma\Theta}{\Sigma_{cc}} \det\left(\left(\begin{array}{c|c} e_d & e_c \end{array} \right) \right) \cdot \det\left(\left(\begin{array}{c|c} e_1 - e_2 & e_c \end{array} \right) \right) + \mathcal{O}(\gamma) \end{aligned}$$

for any state (m_2, t). In particular, even if the "κ" in the dynamic setting are different from their counterparts in the stationary setting, they generate the same flow of surplus.[4]

Alternatively, aggregating (A.48) across the population and recalling the market-clearing condition

$$\mathrm{E}^{\mu(i\theta; m_2, t)} \left[\pi(i\theta; m_2, t) \right] = S_c$$

that holds for any state (m_2, t), yields the ODE

$$\begin{aligned} &\dot{P}_c(m_2,t) - (r + \lambda_a) P_c(m_2,t) \\ &\quad = -\left(m_c - r\gamma \left(\Sigma_{\eta c}(m_2,t) + \Theta\Sigma_{cd} + S_c\Sigma_{cc} \right) + \lambda_a \mathrm{E}^{M(\tilde{m}_2)} \left[P_c(\tilde{m}_2, 0) \right] \right) + \mathcal{O}(\gamma). \end{aligned} \tag{A.49}$$

[4] The "κ"s are defined in Equation (1.21) in the stationary setting and in Equation (1.45) in the dynamic setting.

for the price of the liquid asset. Deriving

$$\Sigma_{\eta c}(m_2, t) = \Sigma_{1c} + \left(\mu_2 + (m_2 - \mu_2)\, e^{-(\lambda_{12}+\lambda_{21})t}\right)(\Sigma_{2c} - \Sigma_{1c})$$

from the type distribution (1.38) and taking as given the value

$$k_0 \triangleq \lambda_a \mathrm{E}^{M(\tilde{m}_2)}\left[P_c(\tilde{m}_2, 0)\right],$$

I can solve the ODE (A.49) in closed form under the no-bubble condition

$$\lim_{T\to\infty} e^{-rT} P_c(m_2, t) = 0.$$

The solution is

$$P_c(h_a, t) = \frac{k_0 + k_1}{r + \lambda_a} + \frac{k_2(m_2)}{r + \lambda_a + \lambda_{12} + \lambda_{21}} e^{-(\lambda_{12}+\lambda_{21})t}, \tag{A.50}$$

with the constants

$$\begin{aligned} k_1 &\triangleq m_c - r\gamma\left(\Sigma_{1c} + \mu_2(\Sigma_{2c} - \Sigma_{1c}) + \Theta\Sigma_{cd} + S_c\Sigma_{cc}\right), \\ k_2(m_2) &\triangleq -r\gamma\left(m_2 - \mu_2\right)(\Sigma_{2c} - \Sigma_{1c}). \end{aligned}$$

I must still find the constant k_0. This is done by solving the linear equation

$$\begin{aligned} & k_0 &=& \lambda_a \mathrm{E}^{M(\tilde{m}_2)}\left[P_c(\tilde{m}_2, 0)\right] \\ \Leftrightarrow\ & k_0 &=& \lambda_a \left(\tfrac{k_0 + k_1}{r + \lambda_a} - r\gamma \tfrac{\mathrm{E}^{M(\tilde{m}_2)}[\tilde{m}_2] - \mu_2}{r + \lambda_a + \lambda_{12} + \lambda_{21}}(\Sigma_{2c} - \Sigma_{1c})\right) \end{aligned}$$

for k_0.

Finally, I characterize how the price P_c of the liquid asset reacts to an aggregate shock that moves the economy from the state (μ_2, t) to the state $(\tilde{m}_2, 0)$. Namely a direct calculation based on Equation (A.50) shows how P_c jumps up when

$$(\Sigma_{2c} - \Sigma_{1c})\left(\tilde{m}_2 - \mu_2(m_2, t)\right) < 0$$

and down when the other inequality holds. □

Proof 50 (Proof of Proposition 14)**.** In the asymptotic case described by the equations (1.47) and (1.48), the HJB equations (1.44) become

$$
\begin{aligned}
& r a(i\theta; m_2, t) \\
= {} & \dot{a}(i\theta; m_2, t) + \kappa(i\theta; m_2, t; \pi(i\theta; m_2, t)) \\
& + \lambda_{i\bar{i}} \left(a\left(\bar{i}\theta; m_2, t\right) - a(i\theta; m_2, t) \right) \\
& + 2\Lambda\mu\left(\bar{i}\bar{\theta}; m_2, t\right) \left[a(i\theta; m_2, t) - P_d(m_2, t)\left(\bar{\theta} - \theta\right) - a(i\theta; m_2, t) \right]^+ \\
& + \lambda_a \mathrm{E}^{M(\tilde{m}_2)} \left[\begin{array}{l} \delta(i; m_2, t; \tilde{m}_2) \cdot \\ \cdot \left(\begin{array}{l} a\left(\bar{i}\theta; \tilde{m}_2, 0\right) + \pi(i\theta; m_2, t)\left(P_c(\tilde{m}_2, 0) - P_c(m_2, t)\right) \\ \quad - a(i\theta; m_2, t) \end{array} \right) \\ + (1 - \delta(i; m_2, t; \tilde{m}_2)) \cdot \\ \cdot \left(\begin{array}{l} a(i\theta; \tilde{m}_2, 0) + \pi(i\theta; m_2, t)\left(P_c(\tilde{m}_2, 0) - P_c(m_2, t)\right) \\ \quad - a(i\theta; m_2, t) \end{array} \right) \end{array} \right], \\
& + o(\gamma)
\end{aligned}
\tag{A.51}
$$

with the optimal holdings "$\pi(i\theta; m_2, t)$" being defined in Lemma 49 for any type $i\theta$ and state (m_2, t). Further, the asymptotic behavior of Equation (1.46) characterizes the bargained price P_d as

$$
P_d(m_2, t) = (a(2\Theta; m_2, t) - a(20; m_2, t)) - \eta_\Theta \left(\begin{array}{l} (a(2\Theta; m_2, t) - a(20; m_2, t)) \\ -(a(1\Theta; m_2, t) - a(10; m_2, t)) \end{array} \right).
$$

Just like in the stationary setting, it is convenient to first work with value function *differences*. I thus define

$$
\Delta_\Theta(m_2, t) \stackrel{\Delta}{=} a(1\Theta; m_2, t) - a(2\Theta; m_2, t) \tag{A.52}
$$

and

$$
\Delta_0(m_2, t) \stackrel{\Delta}{=} a(20; m_2, t) - a(10; m_2, t). \tag{A.53}
$$

Using the HJB equations (A.51) on the right-hand side of the definition (A.52) and rearranging yields

$$
\begin{aligned}
& (r + \lambda_{12} + \lambda_{21} + \lambda_a)\Delta_\Theta(m_2, t) - \dot{\Delta}_\Theta(m_2, t) \\
= {} & \kappa(1\Theta; m_2, t; \pi(i\theta; m_2, t)) - \kappa(2\Theta; m_2, t; \pi(i\theta; m_2, t)) \\
& - 2\Lambda\mu\left(\bar{i}\bar{\theta}; m_2, t\right)(\Delta_\Theta(m_2, t) + \Delta_0(m_2, t)) \\
& + \lambda_a \mathrm{E}^{M(\tilde{m}_2)} \left[\pi(i\theta; m_2, t)\left(P_c(\tilde{m}_2, 0) - P_c(m_2, t)\right)\right] \\
& + \lambda_a \mathrm{E}^{M(\tilde{m}_2)} \left[(1 - \delta(1; m_2, t; \tilde{m}_2) - \delta(2; m_2, t; \tilde{m}_2)\Delta_\Theta(\tilde{m}_2, t)\right] \\
& + o(\gamma)
\end{aligned}
. \tag{A.54}
$$

The same procedure applied to the definition (A.52) yields

$$
\begin{aligned}
&(r+\lambda_{12}+\lambda_{21}+\lambda_a)\,\Delta_0\,(m_2,t)-\dot{\Delta}_0\,(m_2,t)\\
&=\kappa\,(20;m_2,t;\pi(i\theta;m_2,t))-\kappa\,(10;m_2,t;\pi(i\theta;m_2,t))\\
&\quad-2\Lambda\mu\left(\bar{i}\bar{\theta};m_2,t\right)(\Delta_\Theta(m_2,t)+\Delta_0(m_2,t))\\
&\quad+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[\pi(i\theta;m_2,t)\,(P_c\,(\tilde{m}_2,0)-P_c\,(m_2,t))\right]\\
&\quad+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta(1;m_2,t;\tilde{m}_2)-\delta(2;m_2,t;\tilde{m}_2)\,\Delta_\Theta(\tilde{m}_2,t)\right]\\
&\quad+o(\gamma)
\end{aligned}
\quad. \tag{A.55}
$$

Finally, taking the difference of (A.54) and (A.55) and recalling the characterization of $\mathscr{S}$ in Lemma 49 yields the ODE

$$
\begin{aligned}
&\left(r+\lambda_{12}+\lambda_{21}+\lambda_a+2\Lambda\left(\eta_0\mu(1\Theta;m_2,t)+\eta_\Theta\mu(20;m_2,t)\right)\right)\epsilon_{1\Theta}(m_2,t)-\dot{\epsilon}_{1\Theta}(m_2,t)\\
&=\mathscr{S}+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta\,(1;m_2,t;\tilde{m}_2)-\delta\,(2;m_2,t;\tilde{m}_2))\,\epsilon_{1\Theta}\,(\tilde{m}_2,t)\right]+o(\gamma)
\end{aligned}
\tag{A.56}
$$

for the surplus

$$
\begin{aligned}
\epsilon_{1\Theta}(m_2,t) &\overset{\Delta}{=} -\Delta_\Theta(m_2,t)-\Delta_0(m_2,t)\\
&\overset{(\Delta)}{=} a(10;m_2,t)-a(1\Theta;m_2,t)-a(20;m_2,t)+a(2\Theta;m_2,t).
\end{aligned}
$$

Defining

$$
R(m_2,t)\overset{\Delta}{=} r+\lambda_{12}+\lambda_{21}+\lambda_a+2\Lambda\left(\eta_0\mu(1\Theta;m_2,t)+\eta_\Theta\mu(20;m_2,t)\right)
$$

and taking as given the function

$$
F(m_2,t)\overset{\Delta}{=}\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta\,(1;m_2,t;\tilde{m}_2)-\delta\,(2;m_2,t;\tilde{m}_2))\,\epsilon_{1\Theta}\,(\tilde{m}_2,t)\right],
$$

the solution to the ODE (A.56) under the “no-bubble” condition

$$
\lim_{T\to\infty} e^{-rT}\epsilon_{1\Theta}(m_2,t)=0
$$

is

$$
\epsilon(m_2,t)=\mathscr{S}\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}\,\mathrm{d}u+\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}F(m_2,u)\,\mathrm{d}u. \tag{A.57}
$$

Finally, I must still check the existence of $F(m_2,t)$. $F(m_2,t)$ must satisfy

$$\begin{aligned} F(m_2,t) &= \lambda_a \mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta(1;m_2,t;\tilde{m}_2)-\delta(2;m_2,t;\tilde{m}_2))\,\epsilon_{1\Theta}(\tilde{m}_2,t)\right] \\ \Leftrightarrow \quad F(m_2,t) &= \lambda_a \mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta(1;m_2,t;\tilde{m}_2)-\delta(2;m_2,t;\tilde{m}_2))\right]\cdot \\ &\quad \cdot\left(\mathscr{S}\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}\,\mathrm{d}u+\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}F(m_2,u)\,\mathrm{d}u\right). \end{aligned} \tag{A.58}$$

One checks that the right-hand side of this last equality, seen as the image of the function $u \mapsto F(m_2,u)$ by an operator, satisfies the Blackwell's sufficient conditions for a contraction (monotonicity and "discounting"). In particular, there is exactly one solution to the equality (A.58), and this solution must be positive when $\mathscr{S}$ is. Furthermore, this solution is increasing in $\mathscr{S}$ and decreasing in Λ because the right-hand side of (A.58) is.

Given F, Equation (A.57) gives the surplus $\epsilon_{1\Theta}$, and inspection shows that the surplus is also positive when $\mathscr{S}$ is, increasing in $\mathscr{S}$, and decreasing in Λ.

Then, Equations (A.54) and (A.55) uniquely characterize Δ_Θ and Δ_0, respectively. This can be shown by an argument similar to the one characterizing P_c in Lemma 49. Finally, with $\epsilon_{1\Theta}$, Δ_Θ, and Δ_0, four more arguments similar to the one in Lemma 49 uniquely characterize the "$a(i\theta;m_2,t)$"s. □

Proof 51 (Proof of Proposition 15)**.** I assume that I can write

$$\begin{aligned} P_c(m_2,t) &= P_{c,0}(m_2,t)+r\gamma P_{c,1}(m_2,t)+o(\gamma), \\ \pi(i\theta;m_2,t) &= \pi_{c,0}(i\theta;m_2,t)+r\gamma\pi_{c,1}(i\theta;m_2,t)+o(\gamma), \end{aligned} \tag{A.59}$$

for differentiable functions $P_{c,1}$ and $\{\pi_1(i\theta;m_2,t\}_{i\theta}$. Injecting (A.59) into the first order condition (A.47) for the optimal liquid holdings $\pi(i\theta,m_2,t)$ and recalling the characterization of $P_{c,0}$ and $\{\pi_0(i\theta;m_2,t\}_{i\theta}$ in Proposition 49 yields the equation

$$\begin{aligned} 0 &= \dot{P}_{c,1}(m_2,t)-rP_{c,1}(m_2,t)-r\Sigma_{cc}\pi_1(i\theta;m_2,t) \\ &\quad +\lambda_a \mathrm{E}^{M(\tilde{m})}\left[\begin{array}{l} P_{c,1}(\tilde{m}_2,0)-P_{c,1}(m_2,0) \\ -r\left(\begin{array}{l} \pi_0(i\theta;m_2,t)\left(P_{c,0}(\tilde{m}_2,0)-P_{c,0}(m_2,0)\right) \\ +\delta(i;m_2,t)a_0\left(\bar{i}\theta;\tilde{m}_2,0\right) \\ +(1-\delta(i;m_2,t))\,a_0(i\theta;\tilde{m}_2,0) \\ -a_0(i\theta;m_2,0)) \end{array}\right) \\ \quad\cdot\left(P_{c,0}(\tilde{m}_2,0)-P_{c,0}(m_2,0)\right) \end{array}\right]. \end{aligned} \tag{A.60}$$

for $P_{c,1}$ and $\{\pi_1(i\theta;m_2,t\}_{i\theta}$. Now, as

$$S_c = \mathrm{E}^{\mu(i\theta)}\left[\pi(i\theta;m_2,t\right] = \mathrm{E}^{\mu(i\theta)}\left[\pi_0(i\theta;m_2,t\right]$$

it follows that

$$0 = \mathrm{E}^{\mu(i\theta)}\left[\pi_1(i\theta; m_2, t\right].$$

Aggregating Equation (A.60) across the population then yields

$$\begin{aligned} 0 &= \dot{P}_{c,1}(m_2,t) - r P_{c,1}(m_2,t) \\ &\quad + \lambda_a \mathrm{E}^{M(\tilde{m})}\left[\begin{array}{l} P_{c,1}(\tilde{m}_2,0) - P_{c,1}(m_2,0) \\ -r\left(\begin{array}{l} S_c\left(P_{c,0}(\tilde{m}_2,0) - P_{c,0}(m_2,0)\right)^2 \\ + \left(W(\tilde{m}_2,0) - W(m_2,t)\right)\left(P_{c,0}(\tilde{m}_2,0) - P_{c,0}(m_2,0)\right) \end{array}\right) \end{array}\right], \end{aligned} \tag{A.61}$$

with the notation

$$W_0(m_2,t) \stackrel{\Delta}{=} \mathrm{E}^{\mu(i\theta; m_2,t)}\left[a(i\theta; m_2,t)\right] \left(\stackrel{(\Delta)}{=} \mu(m_2,t)\cdot a(m_2,t)\right). \tag{A.62}$$

for the average certainty equivalent across the population. Combining Equation (A.49) with Equation (A.61) then yields

$$\begin{aligned} &\frac{1}{\mathrm{d}t}\left(\frac{\mathrm{E}\left[P_c(m_2,t+\mathrm{d}t)\right|(m_2,t)]}{P_c(m_2,t)} - r\right) \\ &= r\gamma\left(\begin{array}{l} \frac{1}{P_{c,t}}\left(S_c\Sigma_{cc} + \lambda_a \mathrm{E}\left[\left(P_{c,0} - P_{c,t}\right)^2 \middle| (m_2,t)\right]\right) \\ + \frac{1}{P_{c,t}}\left(S_d\Sigma_{cd} + \Sigma_{\eta c}\right) \\ + \lambda_a \mathrm{E}^{m(\tilde{h})}\left[\left(\frac{P_c(\tilde{m}_2,0)}{P_c(m_2,t)} - 1\right)\left(W(\tilde{m}_2,0) - W(m_2,t)\right) \middle| (m_2,t)\right] \end{array}\right) + o(\gamma), \end{aligned} \tag{A.63}$$

which is Expression 1.50 in the statement.

I still have to characterize the sensitivity of the expected returns

$$\frac{1}{\mathrm{d}t}\left(\frac{\mathrm{E}\left[P_c(m_2,t+\mathrm{d}t)\right|(m_2,t)]}{P_c(m_2,t)} - r\right)$$

on the meeting rate Λ. On the right-hand side of (A.63), only the difference

$$W(\tilde{m}_2,0) - W(m_2,t)$$

asymptotically depends on Λ. Hence, I will first looks more carefully at $W(m_2,t)$. It follows from the definition (A.62) of the average certainty equivalent W and the asymptotic HJB equations (A.51) that

$$
\begin{aligned}
& rW(m_2,t) - \mu(m_2,t)\cdot \dot{a}(m_2,t) \\
= {} & \mu(m_2,t)\cdot \kappa\left(m_2,t;\pi(m_2,t)\right) \\
& + \left(\lambda_{12}\mu(1l;m_2,t) - \lambda_{21}\mu(2l;m_2,t)\right)\Delta_0(m_2,t) \\
& + \left(\lambda_{21}\mu(2\Theta;m_2,t) - \lambda_{12}\mu(1\Theta;m_2,t)\right)\Delta_\Theta(m_2,t) \\
& + 2\Lambda\mu\left(20;m_2,t\right)\mu\left(1\Theta;m_2,t\right) \\
& + \lambda_a S_c\left(\mathrm{E}^{M(\tilde{m}_2)}\left[P_c\left(\tilde{m}_2,0\right)\right] - P_c\left(m_2,t\right)\right) \\
& + \lambda_a\left(\mathrm{E}^{M(\tilde{m}_2)}\left[W\left(\tilde{m}_2,0\right)\right] - W\left(m_2,0\right)\right) \\
& + o(\gamma)
\end{aligned}
.
$$

Rearranging then yields the ODE

$$
(r+\lambda_a)\,W(m_2,t) - \dot{W}(m_2,t) = \mu(m_2,t)\cdot\kappa(m_2,t) + \lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[W\left(\tilde{m}_2,0\right)\right]
$$

for $W(m_2,t)$. Under a "no bubble" condition, the unique solution to this ODE is

$$
\begin{aligned}
W(m_2,t) = {} & \int_t^{+\infty} e^{-(r+\lambda_a)(u-t))}\mu\cdot\kappa\,\mathrm{d}u \\
& + \frac{r+\lambda_a}{r}\int_0^{+\infty} e^{-(r+\lambda_a)u}\,\mathrm{E}^{M(\tilde{m}_2)}\left[\mu\left(\tilde{m}_2,u\right)\cdot\kappa\left(\tilde{m}_2,u\right)\right]\,\mathrm{d}u,
\end{aligned}
$$

and I can characterize the quantity of interest as

$$
\begin{aligned}
W\left(\tilde{m}_2,0\right) - W\left(m_2,t\right) = {} & \int_0^{+\infty} e^{-(r+\lambda_a)u}\mu\left(\tilde{m}_2,u\right)\cdot\kappa\left(\tilde{m}_2,u\right)\,\mathrm{d}u \\
& - \int_t^{+\infty} e^{-(r+\lambda_a)(u-t))}\mu\left(m_2,u\right)\cdot\kappa\left(m_2,u\right)\,\mathrm{d}u.
\end{aligned}
\tag{A.64}
$$

Finally, combining Equation (A.64) with the result of Lemma 49 regarding the "$\kappa(i\theta;m_2,t)$"s, and those of Lemma 8 regarding the asymptotic type distribution yields

$$\begin{aligned}
&\partial_{\frac{1}{\Lambda}}\left(W\left(\tilde{m}_2,0\right)-W\left(m_2,t\right)\right)+o\left(\frac{1}{\Lambda}\right)+\mathcal{O}\left(\gamma\right)\\
&=\int_0^{+\infty}e^{-(r+\lambda_a)u}\delta_\mu\left(\tilde{m}_2,u\right)\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}\kappa\left(\tilde{m}_2,u\right)\,\mathrm{d}u\\
&\quad-\int_t^{+\infty}e^{-(r+\lambda_a)(u-t))}\delta_\mu\left(m_2,u\right)\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}\kappa\left(m_2,u\right)\,\mathrm{d}u\\
&=-\mathscr{S}\int_0^{+\infty}e^{-(r+\lambda_a)u}\left(\delta_\mu\left(\tilde{m}_2,u\right)-\delta_\mu\left(m_2,u\right)\right)\,\mathrm{d}u.
\end{aligned}$$

Finally, recalling that

$$\delta_\mu\left(m_2,u\right)=\frac{\lambda_{12}}{2}\frac{\frac{S_d}{\Theta}}{\mu_2\left(m_2,t\right)-\frac{S_d}{\Theta}}$$

is decreasing in $\mu_2\left(m_2,t\right)$ over the support of M_2, I conclude that

$$\partial_{\frac{1}{\Lambda}}\left(W\left(\tilde{m}_2,0\right)-W\left(m_2,t\right)\right)<0$$

exactly when

$$\tilde{m}_2<\mu_2\left(m_2,t\right).$$

Combining this last result with Lemma 49 completes the argument. □

Technical results

Lemma 52. *Consider a smooth map $H:\Omega\to\Omega$ for some $\Omega\subset\mathbb{R}^d$. If for any $i=1,\dots,d$, there exists a $\eta<1$ so that*

$$\sum_{j=1}^{d}\left|\frac{\partial H_i}{\partial x_j}\right|\ \le\ \eta,$$

then H is a contraction in l_∞ and has a unique fixed point.

Proof of Lemma 52. Fix $x_1, x_2 \in \Omega$ and define, for $t \in [0,1]$,

$$x(t) \triangleq x_1 + t(x_2 - x_1).$$

Then, for any $i \in \{1, \dots, d\}$,

$$\begin{aligned}
|H_i(x_2) - H_i(x_1)| &= \left| \int_0^1 \sum_j \frac{\partial H_i}{\partial x_j}(x(t))(x_j^2 - x_j^1)\,\mathrm{d}t \right| \\
&\leq \int_0^1 \sum_j \left| \frac{\partial H_i}{\partial x_j}(x(t)) \right| \left| x_j^2 - x_j^1 \right| \mathrm{d}t \\
&\leq \sum_j \left(\partial_{x_j} H\right) \max_j \left| x_{2j} - x_{1j} \right| \int_0^1 \mathrm{d}t \\
&\leq \eta \| x^2 - x^1 \|_{l_\infty}.
\end{aligned}$$

The last claim follows from the Contraction Mapping Theorem (see [Stokey and Lucas, 1989, Theorem 3.2, p.50]). □

Proposition 53. *Let us consider the system of equations*

$$0 = r\beta_k + \sum_{j \neq k} \kappa_{kj} e^{\beta_k - \beta_j} + c_k \triangleq F_k(\beta), \; k \in \{1, \dots, d\} \tag{A.65}$$

with the unknowns $\beta \equiv (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d$. Then, this system admits a unique solution and this solution is monotone decreasing in the components of K and c.

Proof of Proposition 53. I write β_{-k} for the vector of β without β_k.

First note that there exists a unique smooth function

$$G_k = G_k(\beta_{-k}, K_k, c_k)$$

for which $\beta = G_k(\beta_{-k}, K_k, c_k)$ is the unique solution to

$$r\beta + \sum_{j \neq k} \kappa_{kj} e^{\beta - \beta_j} + c_k = 0.$$

Furthermore, G_k is monotone increasing in the components of β_{-k}, and monotone decreasing in κ_{kj} and c_k for all $j \neq k$.

Then, I show that the functions G_k define a contraction by applying Lemma 52. Namely, I first show that G maps a compact set into itself. Let me choose two real numbers $L < U$, and assume that for any $k \in \{1, \dots, d\}$,

$$\beta_k \in [L, U]^{d-1}.$$

For a given k, let me further define two functions, F_k^L and F_k^U, that bound the function F_k defined in (A.65). Namely,

$$\begin{aligned} r\beta + \sum_{j \neq k} \kappa_{kj} e^{\beta - U} + c_k \stackrel{\Delta}{=} & F_k^L(\beta) \\ \leq & F_k(\beta) \\ \leq & F_k^U(\beta) \stackrel{\Delta}{=} r\beta + \sum_{j \neq k} \kappa_{kj} e^{\beta - L} + c_k. \end{aligned}$$

Now, due to the monotonicity of $F_k\left(\cdot, \beta_{-k}\right)$, if

$$0 \leq F_k^L(U) = rU + \sum_{j \neq k} \kappa_{jk} + c_k \tag{A.66}$$

and

$$0 \geq F_k^U(L) = rL + \sum_{j \neq k} \kappa_{jk} + c_k \tag{A.67}$$

then

$$G_k(\beta_{-k}) \in [L, U].$$

But both (A.66) and (A.67) will hold for all $k \in \{1, \dots, d\}$ as soon as

$$U \geq \max_{k \in \{1, \dots, d\}} \frac{-1}{r} \left(\sum_{j \neq k} \kappa_{jk} + c_k \right)$$

and

$$L \leq \min_{k \in \{1, \dots, d\}} \frac{-1}{r} \left(\sum_{j \neq k} \kappa_{jk} + c_k \right).$$

Now, by the Implicit Function Theorem,

$$\frac{\partial G_k(\beta_{-k})}{\partial \beta_j} = \frac{\kappa_{kj} e^{G_k(\beta_{-k}) - \beta_j}}{r + \sum_{j \neq k} \kappa_{kj} e^{G_k(\beta_{-k}) - \beta_j}},$$

which can be bounded strictly below 1, uniformly in $\beta_{-k} \in [L, U]^{d-1}$, for L and U chosen as above. But then, Lemma 52 ensures the existence and uniqueness of a fixed point on $[L,U]^d$. Finally, as $-L$ and U can be chosen arbitrarily large, the existence and uniqueness on $\mathbb{R}^d$ hold.

Monotonicity follows because

$$\beta^* = \lim_{n\to\infty} G^n(\beta_0)$$

for any fixed β_0 and G is monotone. □

Verification argument

I intend to show that the HJB equations (1.13) actually describe an optimal behaviour.

Being more specific, on the one hand, a given agent with wealth w and type $i\theta$ maximizes

$$V(w,i\theta) \triangleq \max_{\bar{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho t}\left(-e^{-\gamma\bar{c}_s}\right)\,\mathrm{d}s\,\middle|\, w_0 = w, i_0\theta_0 = i\theta\right], \tag{A.68}$$

under the conditions that follow.

- The budget constraint

$$\mathrm{d}w_t = r\,w_t\,\mathrm{d}t - \bar{c}_t\,\mathrm{d}t + \,\mathrm{d}\eta_t + \theta_t\,\mathrm{d}D_{dt} + \pi_t\,(\,\mathrm{d}D_{ct} - rP_c\,\mathrm{d}t) - P_d\,\mathrm{d}\theta_t$$

  holds for a liquid holding process taking values in $[-K,K]$, with K positive and large.[5]

- The price P_d is the outcome of a bargaining with another agent;

- For any $T>0$,

$$\mathrm{E}^{w,i\theta}\left[\int_0^T \left(e^{-\rho u}e^{-r\gamma w_u}\right)^2\,\mathrm{d}u\right] < +\infty \tag{A.69}$$

  and

$$\lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}^{w,i\theta}\left[e^{-\gamma W_T}\right] = 0. \tag{A.70}$$

[5] See footnote 12 for conditions on K.

On the other hand, the HJB equation for the problem above is

$$
\begin{aligned}
\rho V(w, i\theta) = \sup_{\tilde{c},\tilde{\pi}} U(\tilde{c}) \\
&+ \frac{\partial V}{\partial w}(w, i\theta)\left(r w - \tilde{c} + m_\eta + \theta m_d + \tilde{\pi}\,(m_c - r P_c)\right) \\
&+ \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(w, i\theta)\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}^* \\
&+ \lambda_{i\bar{i}}\left(V(w, \bar{i}\theta) - V(w, i\theta)\right) \\
&+ 2\Lambda\,\mathrm{E}^{\mu(b)}\left[\mathbf{1}_{\text{surplus}}\left(V(w - (\bar{\theta} - \theta)P_d, i\bar{\theta})) - V(w, i, \theta)\right)\right],
\end{aligned}
\tag{A.71}
$$

and Proposition 5 shows that there exists a unique solution of the form

$$
\tilde{V}(w, i\theta) = -\exp\left(-r\gamma\,(w + a(i\theta) + \bar{a})\right)
$$

to (A.71). It remains to show that the candidate $\tilde{V}$ is the solution to the problem (A.68). This is the object of the next proposition.

Proposition 54. *If the risk aversion γ is small enough, the function $\tilde{V}$ is the solution to the HJB equations* (A.71) *and the associated consumption and investment strategies are optimal.*

Proof of Proposition 54. My argument comprises four steps.

- Lemma 55 shows that no admissible strategy can achieve an expected utility higher than $\tilde{V}$.
- Lemma 56 shows that the strategy dictated by $\tilde{V}$ is admissible when the risk aversion γ is small enough.
- Lemma 57 shows the strategy dictated by the HJB equations yields an expected utility equal to $\tilde{V}$.

I first show that $\tilde{V}$ represents an upper bound on the attainable expected utilities.

Lemma 55. *If all the agents believe that their value function is given by $\tilde{V}$, then, for any admissible consumption strategy $\tilde{c}$ financed by the trading strategy $\tilde{\pi}$,*

$$
\tilde{V}(w, i\theta) \geq \sup_{\tilde{c}} \mathrm{E}^{w,i\theta}\left[\int_0^\infty e^{-\rho u} U(c_u)\,\mathrm{d}u\right].
$$

Proof. First note that the beliefs regarding the value functions will already fix the outcome of the Nash bargaining, meaning that both the price P_d of the illiquid asset and the cross-sectional distribution of types μ are fixed.

Let me choose an admissible consumption strategy c financed by the trading strategy π, and a time $T > 0$. Recalling the budget constraint,

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\,\mathrm{d}u + e^{-\rho T}\tilde{V}(w_T, i_T\theta_T)\right] \\
=&\mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\,\mathrm{d}u + \tilde{V}(w_0, i_0\theta_0) + \int_0^T \mathrm{d}\left(e^{-\rho u}\tilde{V}(w_u, i_u\theta_u)\right)\right] \\
=&\mathrm{E}\left[\begin{array}{l} \tilde{V}(w_0, i_0\theta_0) + \int_0^T e^{-\rho u} U(c_u)\,\mathrm{d}u \\ +\int_0^T \left(-\rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u)\right)\mathrm{d}u + \int_0^T e^{-\rho u}\,\mathrm{d}\left(\tilde{V}(w_u, i_u\theta_u)\right) \end{array}\right] \\
=&\mathrm{E}\left[\begin{array}{l} \tilde{V}(w_0, i_0\theta_0) \\ +\int_0^T e^{-\rho u} \left(\begin{array}{ll} & U(c_u)\,\mathrm{d}u \\ - & \rho\tilde{V}(w_u, i_u\theta_u)\,\mathrm{d}u \\ + & \frac{\partial\tilde{V}}{\partial w}(w_u, i_u\theta_u)\left(\begin{array}{l} (r w_u - c_u)\,\mathrm{d}u \\ +\,\mathrm{d}e_u \\ +\theta_u\,\mathrm{d}D_{du} \\ +\pi_u\,(\mathrm{d}D_{cu} - rP_c\,\mathrm{d}u) \end{array}\right) \\ + & \frac{1}{2}\frac{\partial^2\tilde{V}}{\partial w^2}(w_u, i_u\theta_u)\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}^*\,\mathrm{d}u \\ + & \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta)\right)\mathrm{d}N_u^i \\ + & \max\left\{0, \begin{array}{c} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{array}\right\}\mathrm{d}N_u^m \end{array}\right) \end{array}\right] \\
=&\mathrm{E}\left[\begin{array}{l} \tilde{V}(w_0, i_0\theta_0) \\ +\int_0^T e^{-\rho u} \left(\begin{array}{ll} & U(c_u)\,\mathrm{d}u \\ - & \rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u)\,\mathrm{d}u \\ + & \frac{\partial\tilde{V}}{\partial w}(w_u, i_u\theta_u)\left(\begin{array}{l} r w_u - c_u \\ +m_e \\ +\theta_u m_d \\ +\pi_u\,(m_c - rP_c) \end{array}\right)\mathrm{d}u \\ + & \frac{\partial\tilde{V}}{\partial w}(w_u, i_u\theta_u)\begin{pmatrix} \alpha_\eta(i_u) \\ \alpha_d(i_u) + \theta_u\sigma_d \\ \alpha_d(i_u) + \pi_u\sigma_c \end{pmatrix}\cdot\begin{pmatrix} \mathrm{d}Z_u \\ \mathrm{d}B_{d,u} \\ \mathrm{d}B_{c,u} \end{pmatrix} \\ + & \frac{1}{2}\frac{\partial^2\tilde{V}}{\partial w^2}(w, i\theta)\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}^*\,\mathrm{d}u \\ + & \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right)\mathrm{d}N_u^i \\ + & \max\left\{0, \begin{array}{c} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{array}\right\}\mathrm{d}N_u^m \end{array}\right) \end{array}\right] \\
\overset{\Delta}{=}&(*),
\end{aligned}
$$

with N^i being the idiosyncratic jump process driving the exposure changes and N^m the jump process defining the meeting times on the OTC market.

Now, defining

$$K_1 \overset{\Delta}{=} (r\gamma)^2 \sup_{\substack{i\theta \\ \tilde{\pi} \in [-K,K]}} e^{-2(a(i\theta)+\bar{a})} \begin{pmatrix} 1 & \theta & \tilde{\pi} \end{pmatrix} \Sigma_i \begin{pmatrix} 1 & \theta & \tilde{\pi} \end{pmatrix}^* \in \mathbb{R},$$

and recalling the admissibility condition (A.70) on (c, π), I may write

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_0^t e^{-\rho u}\frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\begin{pmatrix} \alpha_\eta(i_u) \\ \alpha_d(i_u)+\theta_u\sigma_d \\ \alpha_d(i_u)+\pi_u\sigma_c \end{pmatrix}\cdot\begin{pmatrix} \mathrm{d}Z_u \\ \mathrm{d}B_{d,u} \\ \mathrm{d}B_{c,u} \end{pmatrix}\right)^2\right] \\
=&\mathrm{E}\left[\int_0^t \left(e^{-\rho u}\frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\right)^2 \begin{pmatrix} 1 & \theta_u & \pi_u \end{pmatrix}\Sigma_i\begin{pmatrix} 1 & \theta_u & \pi_u \end{pmatrix}^* \,\mathrm{d}u\right] \\
\leq &K_1\,\mathrm{E}\left[\int_0^t e^{-2(\rho u + r\gamma w_u)}\,\mathrm{d}u\right] \\
<&\infty.
\end{aligned}$$

In particular, in $(*)$, the stochastic integrals against the Brownian motions are true martingales, and their expected values equal zero.

I now turn to the stochastic integrals against the Poisson processes. Keeping in mind the admissibility condition (A.69),

$$\begin{aligned}
&\int_0^t \left|\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right| \mathrm{d}u \\
\leq &\sup_{i\theta}\left|e^{-r\gamma(a(i_u\theta_u)+\bar{a})} - e^{-r\gamma(a(i_u\theta_u)+\bar{a})}\right|\int_0^t e^{-r\gamma w_u}\,\mathrm{d}u \\
<&\infty.
\end{aligned}$$

But then, using a classical result (see, for example, Brémaud [1981][Lemma C4, p.235]),

$$\begin{aligned}
&\mathrm{E}\left[\int_0^T e^{-\rho u}\left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right)\mathrm{d}N_u^i\right] \\
=&\mathrm{E}\left[\int_0^T e^{-\rho u}\lambda_{i_u\bar{i}_u}\left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right)\mathrm{d}u\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathrm{E}\left[\int_0^T e^{-\rho u}\max\left\{0, \begin{matrix} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{matrix}\right\}\mathrm{d}N_u^\theta\right] \\
=&\mathrm{E}\left[\int_0^T e^{-\rho u} 2\lambda\mu(\bar{i}_u\bar{\theta}_u)\max\left\{0, \begin{matrix} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{matrix}\right\}\mathrm{d}u\right].
\end{aligned}$$

I may thus write

$$
\begin{aligned}
(*) &= \mathrm{E}\left[\begin{array}{l}
\tilde{V}(w_0, i_0\theta_0) \\
+\int_0^T e^{-\rho u}\left(\begin{array}{ll}
 & U(c_u) \\
- & \rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u) \\
+ & \frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\left(\begin{array}{l} r w_u - c_u \\ + m_\eta \\ + \theta_u m_d \\ + \pi_u (m_c - r P_c)\end{array}\right) \\
+ & \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial w^2}(w, i\theta)\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}^* \\
+ & \lambda_{i_u\bar{i}_u}\left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right) \\
+ & 2\Lambda\mu\left(\bar{i}_u\bar{\theta}_u\right)\max\left\{0, \begin{array}{c}\tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u)\end{array}\right\}
\end{array}\right) \mathrm{d}u
\end{array}\right] \\
&\le \mathrm{E}\left[\begin{array}{l}
\tilde{V}(w_0, i_0\theta_0) \\
+\int_0^T e^{-\rho u}\sup_{\tilde{c},\tilde{\pi}}\left(\begin{array}{ll}
 & U(\tilde{c}) \\
- & \rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u) \\
+ & \frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\left(\begin{array}{l} r w_u - \tilde{c} \\ + m_\eta \\ + \theta_u m_d \\ + \tilde{\pi} (m_c - r P_c)\end{array}\right) \\
+ & \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial w^2}(w_u, i_u\theta_u)\begin{pmatrix}1 & \theta_u & \tilde{\pi}\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \tilde{\pi}\end{pmatrix}^* \\
+ & \lambda_{i_u\bar{i}_u}\left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right) \\
+ & 2\Lambda\mu\left(\bar{i}_u\bar{\theta}_u\right)\max\left\{0, \begin{array}{c}\tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u)\end{array}\right\}
\end{array}\right) \mathrm{d}u
\end{array}\right] \\
&= \tilde{V}(w_0, i_0\theta_0).
\end{aligned}
$$

Taking things together, this means that, for any $T > 0$,

$$\tilde{V}(w_0, i_0\theta_0) \ge \mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\,\mathrm{d}u\right] + e^{-\rho T}\,\mathrm{E}\left[\tilde{V}(w_T, i_T\theta_T)\right].$$

Letting T become arbitrarily large in this last expression, recalling the admissibility condition (A.70) satisfied by the strategy (c, π), and realizing that the process

$$(a(i_t\theta_t))_{t\ge 0}$$

can only take one of four finite values, yields

$$\tilde{V}(w_0, i_0\theta_0) \ge \lim_{T\to\infty} \mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\,\mathrm{d}u\right] + e^{-\rho T}\,\mathrm{E}\left[\tilde{V}(w_T, i_T\theta_T)\right]$$

$$
\begin{aligned}
&\geq \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(c_u)\,\mathrm{d}u\right] + \lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[-e^{-r\gamma w_T}\right] \sup_{i\theta} e^{-r\gamma(a(i\theta)+\bar{a})} \\
&= \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(c_u)\,\mathrm{d}u\right].
\end{aligned}
$$

As the consumption and trading strategies were arbitrary, this concludes. □

I now propose a condition under which the strategy dictated by the HJB equations is admissible.

Lemma 56. *For γ small enough, the strategy $(\hat{c}_t, \hat{\pi}_t)$ dictated by the optimization in the HJB equation is admissible.*[6]

Proof. The candidate strategy must satisfy two admissibility properties. The first one is (A.69), meaning

$$
\mathrm{E}\left[\int_0^T \left(e^{-\rho u} e^{-r\gamma \hat{w}_u}\right)^2 \mathrm{d}u\right] < \infty.
$$

Now, from Proposition 4, the optimal consumption policy is

$$
\hat{c}(i\theta, w) = r\,(w + a(i\theta) + \bar{a}) - \frac{1}{\gamma}\log(r),
$$

and the resulting wealth dynamics are

$$
\mathrm{d}\hat{w}_t = \left(-r\,(a(i\theta)+\bar{a}) + \frac{1}{\gamma}\log(r)\right)\mathrm{d}t + \mathrm{d}\eta_t + \theta_t\,\mathrm{d}D_{d,t} + \hat{\pi}_t\left(\mathrm{d}D_{d,t} - rP_d\,\mathrm{d}t\right) - P_d\,\mathrm{d}\theta_t.
$$

I may thus write

$$
\begin{aligned}
&\hat{w}_t - w_0 \\
&= \int_0^t \left(-r\,(a(i\theta)+\bar{a}) + \frac{1}{\gamma}\log(r) + m_\eta + \theta_u m_d + \hat{\pi}_u\,(m_c - rP_c)\right)\mathrm{d}u - P_d\,(\theta_T - \theta_0) \\
&\qquad + \int_0^t \begin{pmatrix} \alpha_d(i_u) + \theta_u\sigma_d \\ \alpha_c(i_u) + \pi_u\sigma_c \\ \alpha_\eta(i_u) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}B_{d,u} \\ \mathrm{d}B_{c,u} \\ \mathrm{d}Z_u \end{pmatrix}.
\end{aligned}
$$

In particular, recalling that the Brownian motions and Poisson processes are independent,

[6] I let the risk aversion coefficient go to zero, $\gamma \to 0$, and simultaneously scale up the diffusion coefficients, as described in Equation (1.48).

and defining, for $t \geq 0$,

$$m_t \triangleq \int_0^t \left(-r\left(a(i\theta)+\bar{a}\right) + \frac{1}{\gamma}\log(r) + m_\eta + \theta_u m_d + \hat{\pi}_u \left(m_c - rP_c\right) \right) \mathrm{d}u - P_d \left(\theta_t - \theta_0\right),$$

and

$$s_t^2 \triangleq \int_0^t \begin{pmatrix} 1 & \theta_u & \hat{\pi}(i_u\theta_u) \end{pmatrix} \Sigma_i \begin{pmatrix} 1 \\ \theta_u \\ \hat{\pi}(i_u\theta_u) \end{pmatrix} \mathrm{d}u,$$

I know that the distribution of the wealth conditional on the history of the correlation shocks and OTC trades is

$$\mathscr{L}\left(\hat{w}_t \,|\, (i_u\theta_u)_{0\leq u\leq t}\right) = \mathscr{N}\left(m_t, s_t^2\right).$$

Further, for $t \geq 0$, and defining the two constants

$$K_2 \triangleq \min_{\substack{i\theta \\ \pi\in[-K,K]}} \left\{-ra(i\theta) + \theta m_d + \pi\left(m_c - rP_c\right)\right\}$$

and

$$K_3 \triangleq \sup_{i\theta} \begin{pmatrix} 1 & \theta & \pi(i\theta) \end{pmatrix} \Sigma_i \begin{pmatrix} 1 \\ \theta \\ \pi(i\theta) \end{pmatrix},$$

I can write both

$$m_t \geq t\left(K_2 + \frac{1}{\gamma}\log(r) - r\bar{a} + m_\eta\right) - |P_d|\Delta_\theta,$$

and

$$s_t^2 \leq tK_3.$$

As a result,

$$\begin{aligned}
\mathrm{E}\left[\int_0^T \left(e^{-\rho u} e^{-r\gamma\hat{w}_u}\right)^2 \mathrm{d}u\right] &= \int_0^T e^{-2\rho u}\,\mathrm{E}\left[e^{-2r\gamma\hat{w}_u}\right] \mathrm{d}u \\
&= \int_0^T e^{-2\rho u}\,\mathrm{E}\left[\mathrm{E}\left[e^{-2r\gamma\hat{w}_u}\,\middle|\,(i_v\theta_u)_{0\leq v\leq u}\right]\right] \mathrm{d}u \\
&= \int_0^T e^{-2\rho u}\,\mathrm{E}\left[e^{-2r\gamma m_u + 2(r\gamma)^2 s_u^2}\right] \mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T e^{-2\rho u - 2r\gamma u\left(K_2 + \frac{1}{\gamma}\log(r) - r\bar{a} + m_\eta\right) + 2r\gamma|P_d|\Delta_\theta + 2u(r\gamma)^2 K_3}\, \mathrm{d}u \\
&\leq e^{2r\gamma|P_d|\Delta_\theta} \int_0^T e^{-2u\left(\rho + r\gamma\left(K_2 + \frac{1}{\gamma}\log(r) - r\bar{a} + m_\eta\right) - (r\gamma)^2 K_3\right)}\, \mathrm{d}u \\
&< \infty.
\end{aligned}
$$

I must still show that the candidate policy satisfies the transversality condition (A.70), meaning that

$$
\lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\right] = 0.
$$

The argument is similar to the one in the first part. Namely, for a given $T > 0$,

$$
\begin{aligned}
e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\right] &= e^{-\rho T}\,\mathrm{E}\left[\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\,\middle|\,(i_s\theta_s)_{0\leq s\leq t}\right]\right] \\
&= \mathrm{E}\left[e^{-\rho T - r\gamma m_T + \frac{1}{2}(r\gamma)^2 s_T^2}\right].
\end{aligned}
\tag{A.72}
$$

Now, recalling the choice

$$
\bar{a} \overset{(\Delta)}{=} \frac{1}{r\gamma}\left(-1 + \frac{\rho}{r} + \gamma m_e + \log(r)\right)
$$

in (1.20), the exponent on the right-hand side of (A.72) is

$$
\begin{aligned}
&-\rho T - r\gamma m_T + \frac{1}{2}(r\gamma)^2 s_T^2 \\
&= \int_0^T \begin{pmatrix} -\rho - r\gamma\left(\frac{1}{\gamma}\log(r) - r\left(a(i_u\theta_u) + \bar{a}\right) + m_\eta + \theta_u m_d + \hat{\pi}_u\left(m_c - rP_c\right)\right) \\ +\frac{1}{2}(r\gamma)^2\begin{pmatrix}1 & \theta_u & \hat{\pi}(i_u\theta_u)\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \hat{\pi}(i_u\theta_u)^*\end{pmatrix}\end{pmatrix}\mathrm{d}u \\
&\quad - r\gamma P_d\left(\theta_T - \theta_0\right) \\
&= \int_0^T r\gamma\left(-\frac{1}{\gamma} + ra\left(i_u\theta_u\right) - \kappa(i_u\theta_u)\right)\mathrm{d}u - r\gamma P_d\left(\theta_T - \theta_0\right).
\end{aligned}
\tag{A.73}
$$

Now, recall that the "$\kappa(i\theta)$"s are independent of γ in the asymptotic case defined by the equations (1.47) and (1.48). Hence, for a small enough γ, there exists an $\epsilon > 0$ for which

$$
\int_0^T \left(-\frac{1}{\gamma} + ra(i_u\theta_u) - \kappa(i_u\theta_u)\right)\mathrm{d}u \leq \int_0^T -\epsilon\,\mathrm{d}u = -\epsilon T.
\tag{A.74}
$$

Finally, combining (A.72), (A.73), and (A.74),

$$
0 \leq \lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\right] \leq \lim_{T\to\infty} e^{r\gamma|P_d|\Delta_\theta} e^{-r\gamma\epsilon T} = 0,
$$

as stated. □

The last step is to show that the beliefs are rational. In other words, I must show that the strategy dictated by $\tilde{V}$ and the HJB equations indeed generates an expected utility from consumption equal to $\tilde{V}$.

Lemma 57. *Assuming that γ is small enough, in the sense of Lemma 56, and writing $(\hat{c}, \hat{\pi})$ for the strategy dictated by the HJB equations, then*

$$\tilde{V}(w, i\theta) = \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(\hat{c}_u) \middle| w_0 = w, i_0\theta_0 = i\theta\right].$$

Proof. Thanks to the admissibility of the candidate policy, first, the process

$$\left(\int_0^t e^{-\rho u} U(c_u)\,\mathrm{d}u + e^{-\rho u} \tilde{V}(w_u, i_u\theta_u)\right)_{t\geq 0}$$

is a martingale and, second,

$$\lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[-e^{-r\gamma w_T}\right] = 0.$$

One may then conclude that

$$\tilde{V}(w_0, i_0\theta_0) = \mathrm{E}\left[\int_0^T e^{-\rho u} U(\hat{c}_u)\,\mathrm{d}u\right],$$

by an argument similar to the one in the proof of Lemma 55. □

This concludes the proof of Proposition 54. □

B Proofs for Chapter 2

Proof of Proposition 19. This proof is based on arguments in Gârleanu [2009] and, more specifically, in Section of that article. First, given a trading pattern on the OTC market, we know the type dynamics. And given both the type dynamics and a vector of liquid holdings as described in Assumption 18, the dynamics of the holdings in the liquid asset c are also known.

Second, taking the vector of liquid holdings $\tilde{\Pi}$ and the equilibrium mid-price P_c as given, the value functions must satisfy the HJB equation (2.10). When we let the coefficient of risk-aversion γ go to zero, and scale accordingly the diffusion coefficients, the equation (2.10) becomes the HJB equation for a problem closely related to the original investor's problem. Namely, asymptotically, the equation (2.10) is the HJB equation for a problem in which the value functions are given by

$$a\left(i\theta,\pi;\tilde{\Pi}\right) \stackrel{\Delta}{=} \mathrm{E}^{\tilde{\Pi}}\left[\begin{array}{l} \int_t^{\infty} e^{-r(s-t)}\kappa\left(i_s\theta_s,\pi_s\right)\,\mathrm{d}s \\ \quad -P_d\int_t^{\infty} e^{-r(s-t)}\,\mathrm{d}\theta_s \\ \quad -\int_t^{\infty} e^{-r(s-t)}\left(P_c+q\mathbf{1}_{\pi_s>\pi_{s-}}-q\mathbf{1}_{\pi_s<\pi_{s-}}\right)\mathrm{d}\pi_s \end{array}\left|\begin{array}{lcl} e_t & = & e_i \\ \theta_t & = & \theta \\ \pi_t & = & \pi \end{array}\right.\right] \tag{B.1}$$

and for which there is no control. The definition of the "$\kappa\left(i\theta,\pi\right)$"s is recalled in Equation (2.9) in the main text. The problem (B.1) corresponds to investors who are risk-neutral and have quadratic preferences on the asset holdings.[1] Now, as

$$V\left(w,i\theta,\pi;\tilde{\Pi}\right) = -e^{-r\gamma\left(w+a\left(i\theta,\pi;\tilde{\Pi}\right)+\bar{a}\right)},$$

[1] The formulation (B.1) thus draws an explicit link between two competing modeling of asset pricing with search friction. The competing modeling are, on the one hand, the models based on Duffie et al. [2007], in which utility is induced by consumption from wealth, and, on the other hands, models based on Lagos and Rocheteau [2009], in which utility stems from holding the illiquid asset.

optimizing the value functions

$$V\left(w, i\theta, \pi; \tilde{\Pi}\right)$$

over $\tilde{\Pi}$ is equivalent to finding the vector $\tilde{\Pi}$ that maximizes the functions

$$a\left(i\theta, \pi; \tilde{\Pi}\right)$$

in (B.1). We characterize the optimal holdings Π by looking at deviations from Π. Namely, let us consider the deviation consisting of buying ϵ units of the liquid asset c at time t and going back to the optimal strategy Π at the time of the next trade. When ϵ is small enough, the difference between the value function under Π and under the deviation satisfies

$$\begin{aligned}
&\frac{a_\epsilon\left(i\theta,\pi;\tilde{\Pi}\right) - a\left(i\theta,\pi;\tilde{\Pi}\right)}{\epsilon} \\
&= \frac{-\epsilon\left(P_c+q\right)\mathrm{E}^{\tilde{\Pi}}\left[\begin{array}{r|rcl} \int_t^\tau e^{-r(s-t)}\left(\kappa\left(i_s\theta_s,\pi_s+\epsilon\right)-\kappa\left(i_s\theta_s,\pi_s\right)\right) & e_t &=& e_i \\ -e^{-r\tau}\epsilon\left(P_c+q\mathbf{1}_{\pi_\tau>\pi_{\tau-}}-q\mathbf{1}_{\pi_\tau<\pi_{\tau-}}\right) & \theta_t &=& \theta \\ & \pi_t &=& \pi \end{array}\right]}{\epsilon} \\
&\overset{\epsilon\to 0}{\longrightarrow} -\left(P_c+q\right)+\mathrm{E}^{\tilde{\Pi}}\left[\begin{array}{r|rcl} \int_t^\tau e^{-r(s-t)}\left(\partial_\pi\kappa\left(i_s\theta_s,\pi_s\right)\right) & e_t &=& e_i \\ -e^{-r\tau}\left(P_c+q\mathbf{1}_{\pi_\tau>\pi_{\tau-}}-q\mathbf{1}_{\pi_\tau<\pi_{\tau-}}\right) & \theta_t &=& \theta \\ & \pi_t &=& \pi \end{array}\right].
\end{aligned}$$

For the deviation not to be profitable, this last expression must be non-positive. Further noting that the process $(\pi_s)_s$ is constant over the interval $[t,\tau)$, we have shown the inequality (2.11) in the statement.

If τ is the next jump of a Poisson process with intensity λ_τ, then it has an exponential distribution with parameter λ_τ. Combining the inequality (2.11),

$$\mathrm{E}\left[e^{-rs}\right] = \frac{\lambda_\tau}{r+\lambda_\tau}$$

and

$$\mathrm{E}\left[\int_0^\tau e^{-rs}\,\mathrm{d}s\right] = \frac{1}{r+\lambda_\tau}$$

yields the inequality (2.13).

Starting our argument with a marginal sale of the liquid asset, instead of a marginal purchase, yields the inequalities (2.12) and (2.14).

Finally, if an investor is completing a purchase at time t, meaning that

$$\pi_t > \pi_{t-},$$

the effective current price for both a marginal purchase and a marginal sale is $P_c + q$. As a result, the inequality (2.12), or (2.14) when it applies, must hold with a marginal price of $P_c + q$ instead of $P_c - q$ on the left hand-side. As a result, the inequality (2.11) becomes an equality. A similar argument holds in case of a current sale of the liquid asset. □

Proof of Proposition 20. The proof is constructive. More specifically, the equilibrium is the output of the algorithm that follows.

Algorithm 2: Calculating an equilibrium with transaction costs

---

Data: model parameters

Result: a mapping

$$q \mapsto (\Pi, P_c)(q)$$

associating a vector of holdings Π and a mid-price P_c to any level of transaction costs q

begin

- Initialize q to 0 and initialize Π and P_c to the equilibrium holdings and price in the setting without transaction costs, as described in the first chapter of this dissertation
- Derive the trading pattern p induced by Π
- **while** *not all entries of p are ∅ (no trade at all)* **do**
  - For all the actual trades in p, calculate the threshold for the transaction costs beyond which the first order conditions (2.11) and (2.12) cease to hold
  - set the entry (or entries) in p corresponding to the smallest of these thresholds to ∅
  - Recalculate Π using part 3 of Proposition 19
  - Recalculate P_c with the market clearing condition
- **end**

end

By construction, the output of the algorithm satisfies all the equilibrium conditions. □

Lemma 58. *Assuming that the transaction costs q are small enough, the stationary distribution of types is given by*

$$\mu(1lb) = \frac{2\lambda\mu(1h)\mu(2l)}{\lambda_{12}},$$
$$\mu(1ls) = \mu(2l)\frac{\lambda_{21}}{\lambda_{12}},$$

$$\begin{aligned}
\mu(1hs) &= \mu(1h),\\
\mu(2lb) &= \mu(2l),\\
\mu(2hb) &= \mu(1h)\frac{\lambda_{12}}{\lambda_{21}},\\
\mu(2hs) &= \frac{2\lambda\mu(1h)\mu(2l)}{\lambda_{21}},
\end{aligned}$$

with the masses of the types with two digits being those derived in the first version of the model.

Proof. The proof follows immediately from the flow equations. □

C The TRACE reporting System

The Trade Reporting And Compliance Engine (TRACE) is a program initiated in 2002 by the National Association of Security Dealers (NASD). This program collects and disseminates anonymized bond transaction data. This program is aimed at introducing post-trade transparency on US bond markets, a major OTC market. A number of empirical studies find that TRACE made it possible for all investors to trade at prices closer to the inter-dealer price. TRACE is thus generally considered to have been a positive development, and similar programs have been initiated.[1,2]

However, some recent evidence indicates that TRACE may have reduced trading volumes for certain types of bonds. Hence, on certain bond markets, post-trade transparency moved two popular measures of liquidity in opposite directions. Namely, bid-ask spreads dropped, as did trading volumes. Furthermore, certain bond market participants expressed the view that TRACE had been detrimental to market liquidity. A transparent OTC market, these market participants argue, reduces the incentives to hold inventories and “make the market."[3]

[1] Apparently, the reduction of the bid-ask spreads after the introduction of TRACE was driven by the improved bargaining power of the bond investors. See Goldstein, Hotchkiss, and Sirri [2007], Bessembinder, Maxwell, and Venkataraman [2006], and Edwards, Harris, and Piwowar [2007] for empirical discussing TRACE and bid-ask spreads. Bessembinder and Maxwell [2008] provides a non technical discussion on the same topic, with a number of institutional details. Green, Hollifield, and Schürhoff [2007] discuss bid-ask spreads on the OTC market for municipal bonds.

[2] For example, the Financial Industry Regulatory Authority (FINRA) now collects transaction data for securitized products. See Hollifield, Neklyudov, and Spatt [2012] for an empirical analysis of opacity on the markets for securitized products.

[3] Asquith, Covert, and Pathak [2013] find that TRACE decreased the trading volumes by up to 41.3% for certain categories of bonds. Das et al. [2013] argues that TRACE made bond markets less liquid. Bessembinder and Maxwell [2008] reports complaints by bond market investors about the introduction of TRACE and its adverse effect on liquidity. Finally, bond dealers lobbied against the introduction of TRACE. Their main arguments are summarized in NASD [2006] and largely overlap with the complaints in Bessembinder and Maxwell [2008].

D Proofs for Chapter 3

Proof 59 (Proof of Lemma 24)**.** In the HJB equation (3.5), the first-order condition for the optimization over the consumption rate $\tilde{c}$ is

$$\begin{array}{rrcl} & U'(c) - V_w(w,z) & = & 0 \\ \Leftrightarrow & -\gamma U(c) & = & -\alpha V(w,z) \\ \Leftrightarrow & U(c) & = & \frac{\alpha}{\gamma} V(w,z), \end{array}$$

which is the second statement above. Solving for the optimal consumption now yields

$$\begin{array}{rrcl} & U(c) & = & \frac{\alpha}{\gamma} V(w,z) \\ \Leftrightarrow & -e^{-\gamma c} & = & -\frac{\alpha}{\gamma} e^{-\alpha(w+v(z)+\bar{v})} \\ \Leftrightarrow & c & = & \frac{\alpha}{\gamma}(w + v(z) + \bar{v}) - \frac{1}{\gamma}\log\left(\frac{\alpha}{\gamma}\right). \end{array}$$

Also, the concavity of the utility function $U(\cdot)$ ensures that the solution of the first-order condition is a point of maximum. □

Proof 60 (Proof of Proposition 27)**.** The proof of Proposition 27 is in six steps and proceeds along the lines of Definition 26, the definition of a partial equilibrium.

step (i) We first assume that the function $v(\cdot)$ that characterizes the value functions is quadratic. Namely,

$$v(z) = v_0 + v_1 z + v_2 z^2$$

defines a partial equilibrium. Let us also assume that $v_2 < 0$, meaning that $v(\cdot)$ is strictly concave.

step (ii) We derive the optimal purchasing policy, given $v(\cdot)$. Namely, a z-agent who was offered to trade at the price p solves the optimization

$$\sup_{\tilde{q}} \left\{ v\left(z+\tilde{q}\right) - \tilde{q} p \right\}.$$

The first-order condition yields a unique candidate,

$$q = \left(v'\right)^{(-1)}(p) - z = \frac{p - v_1}{2v_2} - z \triangleq Q(z,p).$$

By the concavity assumption on $v(\cdot)$, this candidate is a point of maximum, and $Q(\cdot,\cdot)$ is the optimal purchase policy.

step (iii) We derive the optimal quoting policy given the purchasing policy $Q(\cdot,\cdot)$ and $v(\cdot)$. Let us consider a z-investor who was contacted by a z_a-investor, accepted to provide a quote, and observes the signal

$$s_a = X z_a + (1 - X)\zeta$$

with

$$X \sim B(1,\tau); \quad \zeta \sim \mu; \quad X,\zeta : \text{independent.} \tag{D.1}$$

The optimal quote solves the maximization

$$\sup_{\tilde{p}} \mathrm{E}^{\mathscr{L}(z_a,s_a)} \left[Q\left(z_a,\tilde{p}\right)\tilde{p} + v\left(z - Q\left(z_a,\tilde{p}\right)\right) \middle| \, s_a \right].$$

The first-order condition for this maximization is

$$\begin{aligned}
& \mathrm{E}^{\mathscr{L}(z_a,s_a)} \left[\begin{array}{l} Q_p\left(z_a,p\right)p + Q\left(z_a,p\right) \\ \quad + v'\left(z - Q\left(z_a,p\right)\right)\left(-Q_p(z_a,p)\right) \end{array} \middle| \, s_a \right] = 0 \\
\Leftrightarrow \quad & \mathrm{E}^{\mathscr{L}(z_a,s_a)} \left[\begin{array}{l} \frac{1}{2v_2}p + \left(\frac{p-v_1}{2v_2} - z_a\right) \\ \quad + \left(v_1 + 2v_2\left(z - \left(\frac{p-v_1}{2v_2} - z_a\right)\right)\right)\left(-\frac{1}{2v_2}\right) \end{array} \middle| \, s_a \right] = 0 \\
\Leftrightarrow \quad & \mathrm{E}^{\mathscr{L}(z_a,s_a)} \left[\frac{1}{2v_2}p + \frac{p-v_1}{2v_2} - z_a - \frac{v_1}{2v_2} - z + \frac{p-v_1}{2v_2} - z_a \middle| \, s_a \right] = 0 \\
\Leftrightarrow \quad & \frac{3}{2v_2}(p - v_1) - z - 2\,\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[z_a | \, s_a\right] = 0.
\end{aligned} \tag{D.2}$$

Now, given the definition of the signal in (D.1) and choosing $s \in \mathbb{R}$, Bayes' rule yields

$$\begin{aligned}
&\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[z_a \mid s_a = s\right]\\
=&\mathrm{P}\left[X=1\right]\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[z_a \mid s_a = s,\, X=1\right]\\
&\quad+\mathrm{P}\left[X=0\right]\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[z_a \mid s_a = s,\, X=0\right]\\
=&\tau\,\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[z_a \mid z_a = s,\, X=1\right]\\
&\quad+(1-\tau)\,\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[z_a \mid \zeta = s,\, X=0\right]\\
=&\tau s+(1-\tau)\,\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[z_a\right]\\
=&\tau s+(1-\tau)\mathcal{M}.
\end{aligned} \tag{D.3}$$

Injecting (D.3) into (D.2) and solving for p yields

$$p = v_1 + 2v_2\left(\frac{1}{3}z + \frac{2}{3}\left(\tau s_a + (1-\tau)\mathcal{M}\right)\right) \triangleq P\left(z, s_a\right).$$

Again, the concavity of $v(\cdot)$ makes this candidate a point of maximum, and $P(\cdot,\cdot)$ is the optimal quoting policy.

step (iv) We derive the expected benefits resulting from providing a quote. Namely, let us assume that a z-investor was asked for a quote by a z_a-investor. If the z-investor accept to issue a quote, she will receive the signal

$$s_a(\omega) = X(\omega)z_a + (1 - X(\omega))\zeta = s \tag{D.4}$$

and the optimal quote will be $P(z,s)$, as defined in the step (iii). As a result, the expected benefits resulting from providing a quote are

$$\begin{aligned}
&\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[\sup_{\tilde{p}}\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[Q\left(z_a,\tilde{p}\right)\tilde{p} + v\left(z - Q\left(z_a,\tilde{p}\right)\right) - v(z)\middle| s_a = s\right]\right]\\
=\ &\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[Q\left(z_a, P(z,s)\right)P(z,s) + v\left(z - Q\left(z_a,\tilde{p}\right)\right) - v_0(z)\middle| s_a = s\right]\right]\\
=\ &-\frac{v_2}{3}\mathrm{E}^{\mathcal{L}(z_a,s_a)}\left[\begin{array}{l}\left(4\mathcal{M}^2(1-\tau)^2 - 3(1-\tau)\left(\mathcal{M}^2+\mathcal{V}\right) - 2\mathcal{M}(1-\tau)z + z^2\right)\\ +2\tau(4\mathcal{M}(1-\tau)-z)s_a\\ +(4\tau-3)\tau s_a^2\end{array}\right]\\
\triangleq\ &(*).
\end{aligned}$$

Given the definition of the signal s_a in (D.4), the law of s_a is μ, the cross-sectional distribution of types.[1] Plugging in the moments of s_a, we can rewrite $(*)$ as

$$(*) = -\frac{v_2}{3}\left(4\mathscr{M}^2\left(1-\tau^2\right)+\left(\mathscr{M}^2+\mathscr{V}\right)\left(4\tau^2-3\right)-2\mathscr{M}z+z^2\right),$$

which is a quadratic expression in z with a positive leading coefficient and a determinant equal to

$$-\frac{4}{9}\left(4\tau^2-3\right)v_2^2\mathscr{V}.$$

In particular, under the assumption (3.11), this determinant is always non-positive and the expected benefits from quoting are always non-negative. In particular,

$$A \overset{(\Delta)}{=} \left\{z : \mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\sup_{\tilde{p}}\mathrm{E}^{\mathscr{L}(z_a,s_a)}\left[\,Q\left(z_a,\tilde{p}\right)\tilde{p}+v\left(z-Q\left(z_a,\tilde{p}\right)\right)\middle|\, s_a\right]\right]\geq v(z)\right\}=\mathbb{R}$$

and, in the third line of the right-hand side of the HJB equation (3.10), taking the positive part has no effect.

step (v) We derive the expected benefits resulting from asking for a quote. Combining the results from steps (ii), (iii), and (iv), we calculate the expected benefits to a z-agent who just

[1] Choosing $x \in \mathbb{R}$,

$$\begin{aligned}
\mathrm{P}[s_a \leq x] &= \mathrm{E}[\mathrm{P}[s_a \leq x | X]] \\
&= \mathrm{P}[X=1]\,\mathrm{E}[\mathrm{P}[s_a \leq x | X=1]] + \mathrm{P}[X=0]\,\mathrm{E}[\mathrm{P}[s_a \leq x | X=0]] \\
&= \tau\,\mathrm{P}[z_a \leq x] + (1-\tau)\,\mathrm{P}[\zeta \leq x] \\
&= \mu((-\infty, x]).
\end{aligned}$$

As a result, $s_a \sim \mu$.

asked another agent for a quote to be

$$
\begin{aligned}
& \mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathbf{1}_{\{z_q\in A\}}\sup_{\tilde q}\left(-\tilde q P\left(z_q,s_z\right)+v\left(z+\tilde q\right)-v(z)\right)\right] \\
=\ & \mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\begin{array}{l} -Q(z_q,P\left(z_q,s_z\right))P\left(z_q,s_z\right) \\ \quad +v\left(z+Q(z_q,P\left(z_q,s_z\right))\right)-v(z)\end{array}\middle| X\right]\right] \\
=\ & \mathrm{P}\left[X=1\right]\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\begin{array}{l} -Q(z_q,P\left(z_q,s_z\right))P\left(z_q,s_z\right) \\ \quad +v\left(z+Q(z_q,P\left(z_q,s_z\right))\right)-v(z)\end{array}\middle| X=1\right] \\
& +\mathrm{P}\left[X=0\right]\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\begin{array}{l} -Q(z_q,P\left(z_q,s_z\right))P\left(z_q,s_z\right) \\ \quad +v\left(z+Q(z_q,P\left(z_q,s_z\right))\right)-v(z)\end{array}\middle| X=0\right], \\
=\ & \tau\,\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\begin{array}{l} -Q(z_q,P\left(z_q,z_a\right))P\left(z_q,z_a\right) \\ \quad +v\left(z+Q(z_q,P\left(z_q,z_a\right))\right)-v(z)\end{array}\right] \\
& +(1-\tau)\,\mathrm{E}^{\mathscr{L}(z_q,s_z)}\left[\begin{array}{l} -Q(z_q,P\left(z_q,\zeta\right))P\left(z_q,\zeta\right) \\ \quad +v\left(z+Q(z_q,P\left(z_q,\zeta\right))\right)-v(z)\end{array}\right] \\
=\ & -\frac{1}{9}v_2\left(\begin{array}{l} \mathscr{M}^2\left(4(\tau-3)\tau^2+9\right)+\mathscr{V}\left(4(1-\tau)\tau^2+1\right) \\ -\mathscr{M}\left(4(\tau-3)\tau^2+9\right)z \\ +\left(4(\tau-3)\tau^2+9\right)z^2\end{array}\right)
\end{aligned}
$$

which is quadratic in the current type z.

step (vi) Using the assumption regarding $v(\cdot)$ in step (i) along with results in steps (iv) and (v), we rewrite the HJB equation (3.10) as

$$
\begin{aligned}
0 =\ & \frac{1}{9}v_2\left(-4\lambda\mathscr{M}^2\left((\tau-3)\tau^2+3\right)+9\sigma_z^2+4\lambda\left((\tau-4)\tau^2+2\right)\mathscr{V}\right)-r\,v_0 \\
& +\left(\mu+\frac{8}{9}\lambda\mathscr{M}\left((\tau-3)\tau^2+3\right)v_2-r\,v_1\right)z \\
& +\left(-\frac{1}{2}\gamma r\sigma^2-\frac{1}{9}v_2\left(4\lambda\left((\tau-3)\tau^2+3\right)+9r\right)\right)z^2.
\end{aligned}
$$

For the equation to hold for any type z, it must be that

$$
\left\{\begin{array}{lcl}
0 & = & \frac{1}{9}v_2\left(-4\lambda\mathscr{M}^2\left((\tau-3)\tau^2+3\right)+9\sigma_z^2+4\lambda\left((\tau-4)\tau^2+2\right)\mathscr{V}\right)-r\,v_0 \\
0 & = & \mu+\frac{8}{9}\lambda\mathscr{M}\left((\tau-3)\tau^2+3\right)v_2-r\,v_1 \\
0 & = & -\frac{1}{2}\gamma r\sigma^2-\frac{1}{9}v_2\left(4\lambda\left((\tau-3)\tau^2+3\right)+9r\right)
\end{array}\right.
$$

This system is linear in the coefficients v_0, v_1, and v_2 and admits the unique solution

$$\begin{cases} v_0 &= \frac{\gamma\sigma^2\left(4\lambda\mathcal{M}^2\left((\tau-3)\tau^2+3\right)-9\sigma_z^2-4\lambda\left((\tau-4)\tau^2+2\right)\mathcal{V}\right)}{8\lambda\left((\tau-3)\tau^2+3\right)+18r} \\ v_1 &= \frac{\mu}{r}-\frac{4\gamma\lambda\mathcal{M}\sigma^2\left((\tau-3)\tau^2+3\right)}{4\lambda\left((\tau-3)\tau^2+3\right)+9r} \\ v_2 &= -\frac{9\gamma r\sigma^2}{8\lambda\left((\tau-3)\tau^2+3\right)+18r} \end{cases}.$$

In particular, for these values of v_0, v_1, and v_2, the function $v(\cdot)$ defined in step (i), the function $Q(\cdot,\cdot)$ defined in step (ii), the function $P(\cdot,\cdot)$ defined in step (iii), and the set $A=\mathbb{R}$ defined in step (iv) satisfy all the conditions in Definition 26 and, thus, define a partial equilibrium of the model.

Proof 61 (Proof of Corollary 29)**.** The type dynamics of a are given by (3.2), meaning by

$$\mathrm{d}z_t = \sigma\,\mathrm{d}B_t + \mathrm{d}\theta_t, \tag{D.5}$$

with the first term being the idiosyncratic variation of the exposure and the second term being the discontinuous changes of exposure induced by trading.

Combining the trading strategy summarized in Section 3.2 and the equilibrium quoting and purchasing strategies stated in Proposition 27, we characterize the possible jumps as follows.

1. When a requests a quote at time t and the signal about her is correct, a's type becomes

   $$z_{t-}+Q\left(z_{t-},P\left(z_q,z_{t-}\right)\right)=\frac{1}{3}z_q+\frac{2}{3}\left(\tau z_{t-}+(1-\tau)\mathcal{M}\right),$$

   with z_q being the type of the quoter.

2. When a requests a quote at time t and the signal about her is uninformative, a's type becomes

   $$z_{t-}+Q\left(z_{t-},P\left(z_q,\zeta_r\right)\right)=\frac{1}{3}z_q+\frac{2}{3}\left(\tau\zeta_r+(1-\tau)\mathcal{M}\right),$$

   with z_q being the type of the quoter and ζ_r being the uninformative signal.

3. When a is asked for a quote at time t and receives an informative signal, a's type becomes

   $$z_{t-}-Q\left(z_r,P\left(z_{t-},z_r\right)\right)=z_r+\frac{2}{3}\left(z_{t-}-\left(\tau z_r+(1-\tau)\mathcal{M}\right)\right),$$

   with z_r being the type of the investor who requested a quote.

4. When a is asked for a quote at time t and receives an uninformative signal, a's type

becomes

$$z_{t-} - Q\left(z_r, P\left(z_{t-}, \zeta_q\right)\right) = z_r + \frac{2}{3}\left(\tau z_{t-} - \left(\tau \zeta_q + (1-\tau)\mathscr{M}\right)\right),$$

with z_r being the type of the investor who requested a quote and ζ_q being the uninformative signal.

Combining (D.5), the four possible jumps we just characterized, and the distributional assumptions in Section 3.2 yields the dynamics (3.12). □

Proof 62 (Proof of Proposition 31)**.** We can derive from Corollary 29 that the generator A of the type dynamics of a given agent is

$$(Af)(z) = \frac{1}{2} f''(z)\sigma_z^2 + \int\int \sum_{j=1}^{4} \lambda_i \left(f\left(\beta_{j\cdot} \begin{pmatrix} 1 \\ z \\ c \\ \zeta \end{pmatrix} \right) - f(z) \right) \mu(z)\mu(\zeta)\, \mathrm{d}z\, \mathrm{d}\zeta,$$

with

$$\lambda = \begin{pmatrix} \lambda\tau & \lambda(1-\tau) & \lambda\tau & \lambda(1-\tau) \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} -\frac{2}{3}\mathscr{M}(1-\tau) & \frac{2}{3} & 1-\frac{2\tau}{3} & 0 \\ -\frac{2}{3}\mathscr{M}(1-\tau) & \frac{2}{3} & 1 & -\frac{2\tau}{3} \\ \frac{2}{3}\mathscr{M}(1-\tau) & \frac{2\tau}{3} & \frac{1}{3} & 0 \\ \frac{2}{3}\mathscr{M}(1-\tau) & 0 & \frac{1}{3} & \frac{2\tau}{3} \end{pmatrix}.$$

We can then derive the stationary Kolmogorov Forward Equation (KFE) from this generator, and taking the Fourier transform of this KFE yields the equation 3.13. Deriving 3.13 once with respect to w, evaluating at $w = 0$, and solving for

$$\hat{\pi}(0) = i\,\mathrm{E}[z] \overset{(\Delta)}{=} i\mathscr{M}$$

shows that the first moment $\mathscr{M}$ can be freely chosen. Deriving 3.13 twice and following the same steps uniquely characterize the second moments of the stationary distribution of types and, as a result, its variance. □

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Education

9/2009 –	**Swiss Federal Institute of Technology Lausanne (EPFL)**, Switzerland Ph.D. candidate in Finance at the **Swiss Finance Institute** Advisor: Semyon Malamud
2/2013 – 6/2013	**Haas School of Business, University of California, Berkeley**, USA visiting scholar Sponsor: Nicolae Gârleanu
10/2006 – 3/2008	**Swiss Federal Institute of Technology Zurich (ETH)**, Switzerland Master of Science in Mathematics
10/2003 – 9/2006	**Swiss Federal Institute of Technology Lausanne (EPFL)**, Switzerland Bachelor of Science in Mathematics
8/2005 – 5/2006	**Carnegie Mellon University,** Pittsburgh, PA USA exchange student, Department of Mathematical Sciences

Research Interests

Theoretical Asset Pricing; Over-the-Counter Markets; Search Frictions; General Equilibrium; Information Asymmetry; Commodity Markets

Working Papers

Equilibrium Asset Pricing with both Liquid and Illiquid Markets (Job Market Paper)
How do the frictions in OTC markets spill over and affect open interests, trading volumes, and risk premia on liquid markets?

Asymmetric Information and Inventory Concerns in Over-the-Counter Markets (with Julien Cujean)
Transparency increases inventory costs and reduces the incentives to provide liquidity on an OTC market.

Equilibrium Commodity Trading (with Emmanuel Leclercq)
Feedback effect of commodity futures: Speculation can be beneficial to the end-users of the commodity even though it increases the volatility of the spot market.

Work in Progress

Competition Between Exchanges and Dealer-Intermediated Markets
General equilibrium model in which investors balance fixed transaction costs and execution uncertainty when they choose a portfolio.

Seminars and Conference Presentations

Equilibrium Asset Pricing with both Liquid and Illiquid Markets

- UC Berkeley (Finance Pre-Seminar at Haas), Copenhagen Business School, Paris Dauphine, EPFL (Brown Bag), University of Geneva, HEC Paris, Université Laval, Norwegian School of Economics, University of Rochester (Simon), Rotterdam School of Management, Stanford GSB
- Erasmus Liquidity Conference, Doctoral Symposium, Rotterdam, 2013

- Econometric Society's North American Summer Meetings, Evanston IL, USA, 2012
- Search and Matching in Financial Markets Workshop, Gerzensee, Switzerland, 2012
- 66th European Meeting of the Econometric Society, Málaga, Spain, 2012
- Fourth European Summer School in Financial Mathematics, Zurich, Switzerland, 2011
- Swiss Doctoral Workshop in Finance, Gerzensee, Switzerland, 2011

Asymmetric Information and Inventory Concerns in Over-the-Counter Markets

- 10th Asset Pricing Retreat, Tilburg (presentation by co-author), The Netherland, 2014

Equilibrium Commodity Trading

- Workshop on Stochastic Games, Equilibrium, and Applications to Energy & Commodities Markets (presentation by co-author), Toronto, Canada, 2013
- Swiss Doctoral Workshop in Finance, Gerzensee, Switzerland, 2013
- Lausanne-Princeton Workshop, Princeton, USA, 2013

Teaching Assistantship

	Swiss Federal Institute of Technology Lausanne (EPFL)
Fall 2013	Advanced Derivatives (for Anders Trolle)
Spring 2012	Investments (for Anders Trolle)
Fall 2010, -11, -12	Stochastic Calculus I (for Semyon Malamud)
	University of Zurich
Fall 2007	Calculus for the Natural Sciences (in German, for Markus Brodmann)

Non-Academic Experience

5/2008- 8/2009	**SunGard BancWare**, Winterthur, Switzerland Quantitative Developer
6/2006- 9/2006	**OptionMetrics, LLC**, New York, NY USA Intern

Refereeing Activity

Mathematical Finance, Journal of Mathematical Economics

Other Information

Citizenship:	Swiss
Programming:	Professional experiences with C#, Matlab, C++
Languages:	French: native; English: fluent; German: fluent

Honors and Awards

Fidessa Doctoral Student Award for outstanding research (2013)

Best Teaching Assistant Award, M.Sc. in Financial Engineering, EPFL (2013)

Swiss Finance Institute Best Discussant Award (2013)

Swiss Finance Institute Best Paper Doctoral Award (2011)

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